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Collapse models with non-white noises: II. Particle-density coupled noises

Stephen L Adler[1] **and Angelo Bassi**[2,3,4]

[1] Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA
[2] Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy
[3] Mathematisches Institut der L.M.U., Theresienstr. 39, 80333 München, Germany
[4] Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Strada Costiera 11, 34014 Trieste, Italy

E-mail: adler@ias.edu, bassi@ts.infn.it and bassi@mathematik.uni-muenchen.de

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Abstract
We continue the analysis of models of spontaneous wavefunction collapse with stochastic dynamics driven by non-white Gaussian noise. We specialize to a model in which a classical 'noise' field, with specified autocorrelator, is coupled to a local nonrelativistic particle density. We derive general results in this model for the rates of density matrix diagonalization and of state vector reduction, and show that (in the absence of decoherence) both processes are governed by essentially the same rate parameters. As an alternative route to our reduction results, we also derive the Fokker–Planck equations that correspond to the initial stochastic Schrödinger equation. For specific models of the noise autocorrelator, including ones motivated by the structure of thermal Green's functions, we discuss the qualitative and quantitative dependence on model parameters, with particular emphasis on possible cosmological sources of the noise field.

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1. Introduction

In an earlier paper [1][5], hereafter referred to as (I), we presented a detailed analysis of stochastic models for state vector collapse driven by Gaussian non-white noise. In particular, we showed that a perturbation expansion in the noise strength parameter $\sqrt{\gamma}$ permits the explicit calculation of consequences of the model, in parallel with standard results obtained by the Itô calculus in the white noise case. In (I) the noise couplings were introduced in generic form, subject to the assumption that the noise correlator has a positive definite structure in

[5] This paper gives extensive references to the stochastic collapse literature, and in particular to prior discussions of models with non-white noise.

the large time limit. As we shall see, this positivity assumption is overly restrictive, and does not apply to the physically interesting case of thermal noise, where the spatial Fourier transform of the noise correlator is oscillatory in time. Our aim in this paper is to specialize the discussion of (I) to the physically interesting case of a particle density-coupled classical noise field, and then within the context of this model, to give a generalized analysis of density matrix diagonalization, state vector reduction and constraints on model parameters. We then turn to the question of whether the noise field postulated in stochastic reduction models can be realized as a cosmological field. (For a section-by-section brief summary of the contents of this paper, the reader should turn to the summary and discussion given in section 7.)

Our starting point in (I) was a diffusion process for the wavefunction in Hilbert space having the form (with $\hbar = 1$, with the constant complex coupling factor ξ introduced in (I) set equal to 1, and with the state vector denoted here by $|\psi\rangle$),

$$\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \left[-\mathrm{i}H + \sqrt{\gamma}\sum_{i=1}^{N} A_i w_i(t) + O\right]|\psi(t)\rangle. \tag{1}$$

Here H is the standard quantum Hamiltonian of the system, A_i are commuting self-adjoint operators, γ is a positive coupling constant and O is a linear operator yet to be defined. The noises $w_i(t)$ are real Gaussian random processes, whose mean and correlation functions are, respectively,

$$\mathbb{E}[w_i(t)] = 0, \qquad \mathbb{E}[w_i(t_1)w_j(t_2)] = D_{ij}(t_1, t_2). \tag{2}$$

We will now specialize the discussion to the case in which the index i is the spatial coordinate $\vec{x}$, and the operator A_i is a particle density $M(\vec{x})$, which, for a many-body system composed of distinguishable particles with couplings m_i and coordinate operators $\vec{q}_i$, is given by

$$M(\vec{x}) = \sum_i m_i \delta^3(\vec{x} - \vec{q}_i). \tag{3}$$

(We have chosen a notation appropriate to the case in which the density M is a mass density, but (3) also describes other forms of coupling to particle densities, such as to the baryon number, lepton number or isospin densities, with m_i the appropriate coupling constants.) An important property of the density operator of (3) is that when integrated over space it reduces to a c-number that commutes with all operators,

$$\int \mathrm{d}^3x\, M(\vec{x}) = \int \mathrm{d}^3x \sum_i m_i \delta^3(\vec{x} - \vec{q}_i) = \sum_i m_i. \tag{4}$$

Hence a noise coupling to the density operator can be permitted to have a nonzero expectation, since this will only contribute a constant term to the effective Hamiltonian on the right of (1). So we will assume that, corresponding to (3), the noises $w_i(t)$ of (I) form a classical noise field, which we shall denote by $\phi(\vec{x}, t)$, with mean and autocorrelation

$$\mathbb{E}[\phi(\vec{x}, t)] = \phi_0, \qquad \mathbb{E}[(\phi(\vec{x}, t_1) - \phi_0))(\phi(\vec{y}, t_2) - \phi_0)] = D(\vec{x} - \vec{y}, t_1 - t_2). \tag{5}$$

Here, in assuming a constant expectation ϕ_0 and in writing the arguments of D, we have built in an assumption of space and time translation invariance; we shall also assume spatial inversion invariance, so that $D(\vec{x}, t) = D(-\vec{x}, t)$. Thus, with this specialization of the noise structure of (I), the diffusion process in Hilbert space of (1) becomes

$$\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \left[-\mathrm{i}H + \sqrt{\gamma}\int \mathrm{d}^3x\, M(\vec{x})\phi(\vec{x}, t) + O\right]|\psi(t)\rangle. \tag{6}$$

In most of what follows, we will neglect the Hamiltonian term in (6), focusing on effects that arise from the action of the stochastic term.

Because it uses real-valued noise, (6) does not preserve the norm of the wavefunction, and this is where the operator O enters. In (I), through detailed calculations that we shall not repeat, we show that O is fixed by the requirements of (i) state vector normalization, and (ii) a linear evolution equation for the density matrix

$$\rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|], \tag{7}$$

which guarantees that superluminal signaling cannot occur. Relation (7) guarantees also the positivity of $\rho(t)$ throughout time. Determining the structure of O leads to three equations from (I), which are exact to order γ, and which when specialized to the case of a density-coupled noise, form the starting point for our analysis here.

The first of the needed equations describes the density matrix time evolution, as given by (53) of (I),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}[H, \rho(t)] + \gamma \int_0^t \mathrm{d}s \int \mathrm{d}^3x \int \mathrm{d}^3y [M(\vec{x})\rho(t)M(\vec{y}, s-t) \\ + M(\vec{y}, s-t)\rho(t)M(\vec{x}) - M(\vec{x})M(\vec{y}, s-t)\rho(t) \\ - \rho(t)M(\vec{y}, s-t)M(\vec{x})]D(\vec{x}-\vec{y}, t-s),\end{aligned} \tag{8}$$

with (see (47) of (I))

$$M(\vec{y}, s-t) = \mathrm{e}^{\mathrm{i}H(s-t)}M(\vec{y})\,\mathrm{e}^{-\mathrm{i}H(s-t)}. \tag{9}$$

When $H = 0$ this simplifies to read (see (19) of (I))

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \gamma \int \mathrm{d}^3x \int \mathrm{d}^3y [M(\vec{x})\rho(t)M(\vec{y}) + M(\vec{y})\rho(t)M(\vec{x}) \\ - M(\vec{x})M(\vec{y})\rho(t) - \rho(t)M(\vec{y})M(\vec{x})]F(\vec{x}-\vec{y}, t),\end{aligned} \tag{10}$$

where we have defined

$$F(\vec{x}-\vec{y}, t) = \int_0^t \mathrm{d}s\, D(\vec{x}-\vec{y}, t-s). \tag{11}$$

To state the second equation, let us define the expectation $\langle O\rangle_t = \langle\psi(t)|O|\psi(t)\rangle$ for any operator O. Then when $H = 0$ the time evolution of the stochastic expectation of the variance $V_A(t) = \langle A^2\rangle_t - \langle A\rangle_t^2$ of any operator A that commutes with the mass density for all $\vec{x}$, given by (23) and (24) of (I), becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[V_A(t)] = -8\gamma \int \mathrm{d}^3x \int \mathrm{d}^3y \mathbb{E}[\langle(M(\vec{x}) - \langle M(\vec{x})\rangle_t)A\rangle_t\langle(M(\vec{y}) \\ - \langle M(\vec{y})\rangle_t)A\rangle_t]F(\vec{x}-\vec{y}, t).\end{aligned} \tag{12}$$

The final equation that we need describes the time evolution of the state vector $|\psi(t)\rangle$, as specified in (40), (51) and (52) of (I), which combined become

$$\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \left[-\mathrm{i}H + \sqrt{\gamma}\int \mathrm{d}^3x[M(\vec{x}) - \langle M(\vec{x})\rangle_t]\phi(\vec{x}, t) + \gamma(B - \langle B\rangle_t)\right]|\psi(t)\rangle, \tag{13}$$

with the self-adjoint operator B given by

$$B = -2\int \mathrm{d}^3x \int \mathrm{d}^3y\, F(\vec{x}-\vec{y}, t)[M(\vec{x}) - \langle M(\vec{x})\rangle_t][M(\vec{y}) - \langle M(\vec{y})\rangle_t]. \tag{14}$$

The alternative form of this equation given in (35) and (37) of (I) differs only by a change of measure for the noise, and makes identical physical predictions.

As is easily checked, an important consequence of the fact that the spatial integral of $M(\vec{x})$ is a c-number (cf (4) is that the noise-field expectation ϕ_0 makes no contribution to the order

$\sqrt{\gamma}$ term in (13), and that a space-independent constant in $F(\vec{x}-\vec{y},t)$ makes no contribution to (12), (10) and (14). That is, we can replace $F(\vec{x}-\vec{y},t)$ by the subtracted function

$$F(\vec{x}-\vec{y},t)-\xi(t), \tag{15}$$

for an arbitrary function $\xi(t)$, with no effect on the equations; only the nonzero spatial Fourier components of $F(\vec{x}-\vec{y},t)$ are significant for our analysis. In particular, for $\xi(t)=F(\vec{0},t)$, this invariance implies that we are free to replace $F(\vec{x}-\vec{y},t)$ by the subtracted function $F(\vec{x}-\vec{y},t)-F(\vec{0},t)$, which has a spatial Fourier transform with improved convergence at small wave numbers.

There has recently been a spirited debate [2] over whether stochastic reduction models can be made relativistically invariant. We remark in this context that the noise coupling of (1) can be obtained in a number of ways as the nonrelativistic limit of relativistically invariant, anti-self-adjoint coupling actions involving scalar, vector or tensor fields. (An anti-self-adjoint action is required to give a real noise term in the Schrödinger equation; we will not attempt here a fundamental justification of this phenomenologically-motivated choice of Hermiticity structure.) When the noise coupling is introduced as the nonrelativistic limit of a relativistic action, relativistic invariance of the stochastic reduction model is broken not by the noise coupling, but by the assumed autocorrelator of the noise field $\phi(\vec{x},t)$. For example, if the noise field has a cosmological origin, its autocorrelator might be expected to refer preferentially to either the Lorentz frame in which the cosmological background radiation is isotropic, or to the galactic rest frame. A topic for future work will be to investigate whether an effective anti-self-adjoint coupling action can arise naturally in a non-equilibrium cosmology, or requires an explicitly non-unitary pre-quantum dynamics.

2. Density matrix diagonalization

We begin our analysis by considering the consequences of (10) for coordinate off-diagonal matrix elements of the density matrix, when the Hamiltonian evolution is neglected. Taking the matrix element of (10) between states $\langle\{\vec{r}_\ell^{\,1}\}|$ and $|\{\vec{r}_\ell^{\,2}\}\rangle$, we get a differential equation for the time dependence of the matrix element of ρ, which can be immediately integrated to give

$$\langle\{\vec{r}_\ell^{\,1}\}|\rho(t)|\{\vec{r}_\ell^{\,2}\}\rangle=\mathrm{e}^{-\Gamma(t)}\langle\{\vec{r}_\ell^{\,1}\}|\rho(0)|\{\vec{r}_\ell^{\,2}\}\rangle, \tag{16}$$

with the integrated rate $\Gamma(t)$ given by

$$\Gamma(t)=\gamma\int\mathrm{d}^3x\int\mathrm{d}^3y\int_0^t\mathrm{d}s\,F(\vec{x}-\vec{y},s)[m_1(\vec{x})-m_2(\vec{x})][m_1(\vec{y})-m_2(\vec{y})], \tag{17}$$

where $m_{1,2}$ are the eigenvalues of the operator $M(\vec{x})$ when acting on the respective states $|\{\vec{r}_\ell^{\,1,2}\}\rangle$,

$$m_1(\vec{x})=\sum_i m_i\delta^3\big(\vec{x}-\vec{r}_i^{\,1}\big),\qquad m_2(\vec{x})=\sum_i m_i\delta^3\big(\vec{x}-\vec{r}_i^{\,2}\big). \tag{18}$$

Substituting (18) into (17) and carrying out the $\vec{x}$ and $\vec{y}$ integrals using the delta functions, we obtain

$$\begin{aligned}\Gamma(t)=\gamma\int_0^t\mathrm{d}s\sum_i\sum_j m_im_j\big[&F\big(\vec{r}_i^{\,1}-\vec{r}_j^{\,1},s\big)+F\big(\vec{r}_i^{\,2}-\vec{r}_j^{\,2},s\big)\\&-F\big(\vec{r}_i^{\,1}-\vec{r}_j^{\,2},s\big)-F\big(\vec{r}_i^{\,2}-\vec{r}_j^{\,1},s\big)\big].\end{aligned} \tag{19}$$

We now review a number of useful features of this formula (many of which, in a slightly different notation, are familiar from the stochastic reduction literature). First of all, as already pointed out in section 1, (19) is unchanged when we replace $F(\vec{r}, s)$ by the subtracted function $F(\vec{r}, s) - F(\vec{0}, s)$. Second, suppose that $\vec{r}_I^1 = \vec{r}_I^2 = \vec{r}_I$ for some particle with index I. Then the contribution of this particle to the double sum in (19) is

$$2m_I^2[F(\vec{0}, s) - F(\vec{0}, s)] + 2m_I \sum_{j \neq I} m_j \big[F\big(\vec{r}_I - \vec{r}_j^1, s\big) + F\big(\vec{r}_I - \vec{r}_j^2, s\big) - F\big(\vec{r}_I - \vec{r}_j^2, s\big) - F\big(\vec{r}_I - \vec{r}_j^1, s\big)\big] = 0. \tag{20}$$

So only particles that have different coordinates in the groups 1 and 2 contribute to the sum.

Third, suppose that for large separations $\vec{r}$, relative to some correlation scale r_C, the function $F(\vec{r}, s)$ asymptotically approaches a constant (which can be zero or nonzero). Then if there are two particles I, J such that $\vec{r}_I^{1,2} - \vec{r}_J^{1,2}$ are all large enough relative to r_C to be in the asymptotic regime for F, the cross terms in the double sum linking these two particles do not contribute. This means that if the particles form a set of K widely spaced bunches on the scale of r_C, with the particles of group 2 displaced with respect to those of group 1 by distances of order r_C, the formula for $\Gamma(t)$ splits into a sum

$$\Gamma(t) = \sum_{k=1}^{K} \Gamma^k(t), \tag{21}$$

with $\Gamma^k(t)$ computed entirely within the kth bunch.

Fourth, let us take group 1 to be a collection of particles that are very closely spaced on the scale of r_C, and suppose that the particles of group 2 are all displaced by a common vector $\vec{R}$ with respect to those of group 1. In this case, $\Gamma(t)$ is approximated by the formula

$$\Gamma(t) \simeq 2\gamma \int_0^t \mathrm{d}s \left(\sum_i m_i\right)^2 [F(\vec{0}, s) - F(\vec{R}, s)], \tag{22}$$

which is the formula that would be obtained if there were only one particle of mass $\sum_i m_i$ at the center of mass of the group. The above formulae display the amplification mechanism typical of collapse models: when particles interact to form a macro-object, the collapses on the single particles add up in such a way that the center of mass of the object collapses each time a single particle does. This is the reason why these models can account both for the quantum properties of microscopic systems and for the classical properties of macroscopic objects.

Fifth, let us again take group 1 to be a collection of particles that are very closely spaced on the scale of r_C, but now suppose that the particles of group 2 are displaced by random amounts, with an average magnitude of displacement R with respect to those of group 1. When the function $F(\vec{r}, s)$ only depends on the magnitude $|\vec{r}|$ of the displacement vector, so that $F(\vec{r}, s) = F[|\vec{r}|, s]$ then $\Gamma(t)$ is approximated by the formula

$$\Gamma(t) \simeq \gamma \int_0^t \mathrm{d}s \left(\sum_i m_i\right)^2 \big[F(0, s) + \langle\!\langle F\big(\big|\vec{r}_i^2 - \vec{r}_j^2\big|, s\big)\rangle\!\rangle_N - 2\langle\!\langle F\big(\big|\vec{r}_i^1 - \vec{r}_j^2\big|, s\big)\rangle\!\rangle_N\big], \tag{23}$$

where $\langle\!\langle \cdots \rangle\!\rangle_N$ denotes the average over the ensemble of particles; when $R > r_C$ (23) is further approximated by

$$\Gamma(t) \simeq \gamma \int_0^t \mathrm{d}s \left(\sum_i m_i\right)^2 [F(0, s) - F(R, s)], \tag{24}$$

which is one half of the $\Gamma(t)$ given by the center-of-mass formula (22) for the corresponding magnitude of R.

Finally, in many cases of interest $F(\vec{x}-\vec{y},s)$ can be written as a sum or integral over factors referring to $\vec{x}$ and $\vec{y}$ separately,

$$F(\vec{x}-\vec{y},s)=\sum_{\alpha}\mathcal{F}(\alpha,s)f(\alpha,\vec{x})f(\alpha,\vec{y}), \tag{25}$$

with $\mathcal{F}$ an appropriate weighting function, and α a shorthand for any combination of discrete and continuous variables. Substitution of (25) into (17) gives

$$\Gamma(t)=\gamma\int_0^t \mathrm{d}s\sum_{\alpha}\mathcal{F}(\alpha,s)g\big(\alpha,\{\vec{r}_\ell^{\,1}\},\{\vec{r}_\ell^{\,2}\}\big)^2, \tag{26}$$

with

$$g\big(\alpha,\{\vec{r}_\ell^{\,1}\},\{\vec{r}_\ell^{\,2}\}\big)=\int \mathrm{d}^3x\, f(\alpha,\vec{x})[m_1(\vec{x})-m_2(\vec{x})]=\sum_i m_i\big[f\big(\alpha,\vec{r}_i^{\,1}\big)-f\big(\alpha,\vec{r}_i^{\,2}\big)\big]. \tag{27}$$

3. State vector reduction

We proceed next to apply (12) to the problem of reduction of a state vector constructed as a superposition of position eigenstates $\big|\{\vec{r}_\ell^{\,j}\}\big\rangle$, $j=1,\ldots,N$. Our first step is to rewrite (12) in a more useful form by setting $A=B+C$, with B and C operators that commute with each other and with the mass density, and subtracting off (12) as written for B and C alone, which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\langle BC\rangle_t-\langle B\rangle_t\langle C\rangle_t]=-8\gamma\int \mathrm{d}^3x\int \mathrm{d}^3y\,\mathbb{E}[\langle (M(\vec{x}) \\ -\langle M(\vec{x})\rangle_t)B\rangle_t\langle (M(\vec{y})-\langle M(\vec{y})\rangle_t)C\rangle_t]F(\vec{x}-\vec{y},t). \tag{28}$$

Using the fact that B and C can be arbitrary operator functions of the particle coordinate operators, by making the specific choices $B=\prod_i\delta^3(\vec{u}_i-\vec{q}_i)$ and $C=\prod_i\delta^3(\vec{w}_i-\vec{q}_i)$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\left(\prod_i\delta^3(\vec{w}_i-\vec{u}_i)\right)|\psi(\{\vec{w}_\ell\})|^2-|\psi(\{\vec{u}_\ell\})|^2|\psi(\{\vec{w}_\ell\})|^2\right]\\ &\quad=-8\gamma\int \mathrm{d}^3x\mathrm{d}^3y\,\mathbb{E}\Bigg[|\psi(\{\vec{u}_\ell\})|^2|\psi(\{\vec{w}_\ell\})|^2\sum_j m_j[\delta^3(\vec{x}-\vec{u}_j)\\ &\qquad-|\hat{\psi}_j(\vec{x})|^2]\sum_k m_k[\delta^3(\vec{y}-\vec{w}_k)-|\hat{\psi}_k(\vec{y})|^2]\Bigg]F(\vec{x}-\vec{y},t),\end{aligned} \tag{29}$$

where we have introduced the definition

$$|\hat{\psi}_j(\vec{z}_j)|^2=\left(\prod_{i\neq j}\int \mathrm{d}^3z_i\right)|\psi(\{\vec{z}_\ell\})|^2,\qquad \int \mathrm{d}^3z_j|\hat{\psi}(\vec{z}_j)|^2=1. \tag{30}$$

Let us now specialize (29) to the case of a wavefunction which is the superposition of N distinct localized groups of particles, by writing

$$\begin{aligned}\psi(\{\vec{z}_\ell\})&=\langle\{\vec{z}_\ell\}|\psi(t)\rangle=\sum_{J=1}^N\alpha_J\prod_\ell\delta^3\big(\vec{z}_\ell-\vec{r}_\ell^{\,J}\big)^{1/2},\\ |\psi(\{\vec{z}_\ell\})|^2&=\sum_{J=1}^N p_J\prod_\ell\delta^3\big(\vec{z}_\ell-\vec{r}_\ell^{\,J}\big),\end{aligned} \tag{31}$$

with $p_J = |\alpha_J|^2$ and with normalization of the wavefunction implying that $\sum_J p_J = 1$. (By the square root of a delta function, we mean a Gaussian wave packet which is sharply localized, with a modulus squared that integrates to unity.) Substituting (31) into (29), and integrating in $d\{\vec{w}_\ell\}$ around $\{\vec{r}_\ell^{\,L}\}$ and in $d\{\vec{u}_\ell\}$ around $\{\vec{r}_\ell^{\,M}\}$, we get an equation for the time evolution of the occupation probabilities p_J of the corresponding states $\prod_\ell \delta^3\big(\vec{z}_\ell - \vec{r}_\ell^{\,J}\big)^{1/2}$ with label J that appear in the superposition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\delta_{ML}p_M - p_M p_L] = -8\gamma\mathbb{E}\Bigg[p_M p_L \sum_j \sum_k m_j m_k \Bigg\{F\big(\vec{r}_j^{\,L} - \vec{r}_k^{\,M}, t\big) + \sum_{R,S} p_R p_S F\big(\vec{r}_j^{\,R} - \vec{r}_k^{\,S}, t\big) - \sum_R p_R F\big(\vec{r}_j^{\,R} - \vec{r}_k^{\,M}, t\big) - \sum_S p_S F\big(\vec{r}_j^{\,L} - \vec{r}_k^{\,S}, t\big)\Bigg\}\Bigg]. \tag{32}$$

Specializing this further to the two group case with $N = 2$, taking $M = L = 1$ and doing some algebraic rearrangement using the fact that the sum of the probabilities is $p_1 + p_2 = 1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[p_1 p_2] = -8\gamma\mathbb{E}\big[p_1^2 p_2^2\big]\sum_j\sum_k m_j m_k\big\{F\big(\vec{r}_j^{\,1} - \vec{r}_k^{\,1}, t\big) + F\big(\vec{r}_j^{\,2} - \vec{r}_k^{\,2}, t\big) - F\big(\vec{r}_j^{\,1} - \vec{r}_k^{\,2}, t\big) - F\big(\vec{r}_j^{\,2} - \vec{r}_k^{\,1}, t\big)\big\}. \tag{33}$$

We can now use (33) to derive upper and lower bounds to the reduction rate, as follows. To obtain an upper bound, we use the inequality

$$\mathbb{E}\big[p_1^2 p_2^2\big] \geqslant \mathbb{E}[p_1 p_2]^2, \tag{34}$$

and the assumption that the integrand of $\Gamma(t)$ in (19) is positive for all s, to rewrite (33) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[p_1 p_2] \leqslant -8\gamma\mathbb{E}[p_1 p_2]^2\sum_j\sum_k m_j m_k\big\{F\big(\vec{r}_j^{\,1} - \vec{r}_k^{\,1}, t\big) + F\big(\vec{r}_j^{\,2} - \vec{r}_k^{\,2}, t\big) - F\big(\vec{r}_j^{\,1} - \vec{r}_k^{\,2}, t\big) - F\big(\vec{r}_j^{\,2} - \vec{r}_k^{\,1}, t\big)\big\}, \tag{35}$$

giving a differential inequality that can be integrated to give an upper bound on the reduction rate

$$\mathbb{E}[p_1(t)p_2(t)] \leqslant \frac{\mathbb{E}[p_1(0)p_2(0)]}{1 + 8\Gamma(t)}. \tag{36}$$

To get a lower bound, we note that since the probabilities p_1 and p_2 obey $p_1 + p_2 = 1$, we have $p_1 p_2 = p_1(1 - p_1) \leqslant 1/4$, and so

$$\mathbb{E}\big[p_1^2 p_2^2\big] \leqslant \mathbb{E}[p_1 p_2]/4. \tag{37}$$

Again assuming that the integrand of (19) is positive for all s, this gives a differential inequality that can be integrated to give the lower bound

$$\mathbb{E}[p_1(t)p_2(t)] \geqslant \mathbb{E}[p_1(0)p_2(0)]\exp[-2\Gamma(t)]. \tag{38}$$

Thus we see that in our model of a Schrödinger equation modified solely by a real noise process, the upper and lower bounds on the reduction rate involve (under the uniform positivity assumption) the same integrated rate function $\Gamma(t)$ as appears in the decay of the off-diagonal density matrix element $\langle\{\vec{r}_\ell^{\,1}\}|\rho(t)|\{\vec{r}_\ell^{\,2}\}\rangle$. Of course, in realistic applications, the rate for density matrix diagonalization is expected to receive much larger contributions from decoherence processes, which can be modeled by imaginary noise terms in the Schrödinger equation that do not contribute to state vector reduction. Although the upper and lower bounds are governed by the same integrated rate function, they have very different functional dependencies: the upper bound depends on the inverse of $\Gamma(t)$, whereas the lower bound is a negative exponential in $\Gamma(t)$. Solvable models [3, 4] and appendix D show that in fact the actual decay of the variance is exponential, rather than power law, indicating that the lower bound of (38) gives the better estimate[6].

In the general N group case, although we have not derived rigorous bounds, we can get estimates similar to the two-group case by setting $M = L$ in (32), giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[p_L(1-p_L)] = -8\gamma\mathbb{E}\left[p_L^2\sum_j\sum_k m_j m_k\left\{F\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,L},t\right)+\sum_{R,S}p_Rp_SF\left(\vec{r}_j^{\,R}-\vec{r}_k^{\,S},t\right)\right.\right.$$
$$\left.\left.-\sum_R p_RF\left(\vec{r}_j^{\,R}-\vec{r}_k^{\,L},t\right)-\sum_S p_SF\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,S},t\right)\right\}\right]. \tag{39}$$

Suppose now that the stochastic process brings the probabilities close to a corner of their domain, where for some $M \neq L$ the probability p_M is close to unity, and thus all the other probabilities are small. The right-hand side of (39) then contains terms of second degree in small quantities, given by selecting the terms with $R = M$ and $S = M$ in the sums, plus remaining terms that are third degree in small quantities. The second-degree terms contribute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[p_L] \simeq -8\gamma\mathbb{E}\left[p_L^2\right]\sum_j\sum_k m_jm_k\left\{F\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,L},t\right)+F\left(\vec{r}_j^{\,M}-\vec{r}_k^{\,M},t\right)\right.$$
$$\left.-F\left(\vec{r}_j^{\,M}-\vec{r}_k^{\,L},t\right)-F\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,M},t\right)\right\}, \tag{40}$$

which has a structure similar to (33) for the two-group case. Using the inequality

$$\mathbb{E}\left[p_L^2\right] \geqslant \mathbb{E}[p_L]^2, \tag{41}$$

defining $\Gamma^{LM}(t)$ by

$$\Gamma^{LM}(t) = \gamma\int_0^t \mathrm{d}s\sum_j\sum_k m_jm_k\left[F\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,L},s\right)+F\left(\vec{r}_j^{\,M}-\vec{r}_k^{\,M},s\right)\right.$$
$$\left.-F\left(\vec{r}_j^{\,L}-\vec{r}_k^{\,M},s\right)-F\left(\vec{r}_j^{\,M}-\vec{r}_k^{\,L},s\right)\right], \tag{42}$$

and assuming positivity of the integrand of (42), we get a differential inequality that can be integrated to give an upper bound on the decay rate,

$$\mathbb{E}[p_L(t)] \leqslant \frac{\mathbb{E}[p_L(0)]}{1+8\Gamma^{LM}(t)}. \tag{43}$$

Similarly, from (40) we can also get a lower bound on the decay rate,

$$\mathbb{E}[p_L(t)] \geqslant \mathbb{E}[p_L(0)]\exp[-2\Gamma^{LM}(t)]. \tag{44}$$

[6] In the example calculated in appendix D, the actual variance decay is $\sim\mathrm{e}^{-\Gamma(t)}$. A simple example shows how an exponential decay of the variance can agree with the inequalities used to get the upper and lower bounds. If $\mathbb{E}[p_1(t)p_2(t)] \simeq \mathbb{E}[p_1(0)p_2(0)]\exp(-\Gamma(t))$, then $(\mathrm{d}/\mathrm{d}t)\mathbb{E}[p_1(t)p_2(t)] = -\Gamma'(t)\mathbb{E}[p_1(t)p_2(t)]$, whereas (33) implies that $(\mathrm{d}/\mathrm{d}t)\mathbb{E}[p_1(t)p_2(t)] = -8\Gamma'(t)\mathbb{E}[p_1^2(t)p_2^2(t)]$, and so we must have $\mathbb{E}[p_1(t)p_(t)] = 8\mathbb{E}[p_1^2p_2^2]$. Suppose now that $p_1(t)p_2(t) = 0$ with probability $1-\exp(-\Gamma(t))$, and $p_1(t)p_2(t) = 1/8$ with probability $\exp(-\Gamma(t))$. We then have $\mathbb{E}[p_1(t)p_2(t)] = 8\mathbb{E}[p_1^2(t)p_2^2(t)] = (1/8)\exp(-\Gamma(t))$. However, $\mathbb{E}[p_1(t)p_2(t)]^2 = (1/64)\exp(-2\Gamma(t))$, which for large $\Gamma(t)$ is much smaller than $\mathbb{E}[p_1^2(t)p_2^2(t)] = (1/64)\exp(-\Gamma(t))$, and so the inequality of (34), which was used to get the upper bound, is far from being saturated, while by construction, the inequality of (37), which was used to get the lower bound, is saturated.

Thus, near the corner where $p_M \simeq 1$, the other p_L decay to zero, with the slowest rate of decrease corresponding to the smallest value of Γ^{LM} for $L \neq M$.

To conclude this section, we note that when $F(\vec{x} - \vec{y}, t)$ has the factorized form given in (25), then (32) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\delta_{ML} p_M - p_M p_L] = -8\gamma \mathbb{E}\left[p_M p_L \sum_\alpha \mathcal{F}(\alpha, t) \sum_j m_j \left[f\left(\alpha, \vec{r}_j^{\,L}\right) - \sum_R p_R f\left(\alpha, \vec{r}_j^{\,R}\right)\right] \right. \\ \left. \times \sum_k m_k \left[f\left(\alpha, \vec{r}_k^{\,M}\right) - \sum_S p_S f\left(\alpha, \vec{r}_k^{\,S}\right)\right]\right], \tag{45}$$

while (42) becomes

$$\Gamma^{LM}(t) = \gamma \int_0^t \mathrm{d}s \sum_\alpha \mathcal{F}(\alpha, t) \left[\sum_i m_i \left(f\left(\alpha, \vec{r}_i^{\,L}\right) - f\left(\alpha, \vec{r}_i^{\,M}\right)\right)\right]^2 \\ = \gamma \int_0^t \mathrm{d}s \sum_\alpha \mathcal{F}(\alpha, t) g\left(\alpha, \left\{\vec{r}_\ell^{\,L}\right\}, \left\{\vec{r}_\ell^{\,M}\right\}\right)^2, \tag{46}$$

and $\Gamma(t) = \Gamma^{12}(t)$ is the specialization of this formula to $L = 1$, $M = 2$.

4. The Fokker–Planck equation

As a complement to the methods used in the preceding sections, we derive the Fokker–Planck equation for the non-white noise model, and use it to rederive (32). We again restrict ourselves to the case when the Hamiltonian H is zero, which allows all equations to be diagonalized in coordinate representation. Starting from (13) and substituting

$$|\psi(t)\rangle = \sum_{L=1}^{N} \alpha_L \left|\left\{\vec{r}_\ell^{\,L}\right\}\right\rangle, \tag{47}$$

with $\left|\left\{\vec{r}_\ell^{\,L}\right\}\right\rangle$ sharply localized wave-packet states (cf (31)), we find that the coefficients α_L obey the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_L = \alpha_L X_L, \tag{48}$$

with X_L given by

$$X_L = \sqrt{\gamma} \sum_i m_i \left[\phi\left(\vec{r}_i^{\,L}, t\right) - \sum_R p_R \phi\left(\vec{r}_i^{\,R}, t\right)\right] \\ - \gamma \sum_i \sum_j 2 m_i m_j \left[F\left(\vec{r}_i^{\,L} - \vec{r}_j^{\,L}, t\right) + 2 \sum_{R,S} p_R p_S F\left(\vec{r}_i^{\,R} - \vec{r}_j^{\,S}, t\right) \right. \\ \left. - 2 \sum_R p_R F\left(\vec{r}_i^{\,L} - \vec{r}_j^{\,R}, t\right) - \sum_R p_R F\left(\vec{r}_i^{\,R} - \vec{r}_j^{\,R}, t\right)\right]. \tag{49}$$

Since X_L is real, and $p_L = \alpha_L^* \alpha_L$, we correspondingly have

$$\frac{\mathrm{d}}{\mathrm{d}t} p_L = 2 p_L X_L. \tag{50}$$

In order to derive the Fokker–Planck equation, we have to evaluate $\mathbb{E}[(\mathrm{d}/\mathrm{d}t) f(\{p_R\})]$ for an arbitrary function f of the set of probabilities $\{p_R\}$, keeping terms through order γ. On using the chain rule we have

$$\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(\{p_R\})\right] = \mathbb{E}\left[\sum_S \frac{\partial f(\{p_R\})}{\partial p_S}\frac{\mathrm{d}}{\mathrm{d}t} p_S\right]. \tag{51}$$

On substituting (50) and (49) for $(\mathrm{d}/\mathrm{d}t)p_S$, we encounter two types of terms. Terms of the form

$$-\gamma\sum_S \mathbb{E}\left[\frac{\partial f(\{p_R\})}{\partial p_S} A_S\right] \tag{52}$$

can be read off directly from the term proportional to $-\gamma$ in X_S, while terms of type

$$\sqrt{\gamma}\sum_{i,S,L}\mathbb{E}\left[\frac{\partial f(\{p_R\})}{\partial p_S} B_{SiL}\phi\left(\vec{r}_i^{\,L}, t\right)\right] \tag{53}$$

are evaluated using the Furutsu–Novikov formula, which approximated by using $p_R(s) = p_R(t) + O(\sqrt{\gamma})$, takes the form

$$\mathbb{E}\left[\frac{\partial f(\{p_R\})}{\partial p_S} B_{SiL}\phi\left(\vec{r}_i^{\,L}, t\right)\right] = \sum_{j,S,T} F\left(\vec{r}_i^{\,L} - \vec{r}_j^{\,S}, t\right)\mathbb{E}\left[\frac{\partial}{\partial p_T}\left(\frac{\partial f(\{p_R\})}{\partial p_S} B_{SiL}\right)\frac{\partial p_T}{\partial\phi\left(\vec{r}_j^{\,S}, t\right)}\right]. \tag{54}$$

The needed derivative of p_T can be read off directly from the $\sqrt{\gamma}$ term in X_T; substituting this, and doing much algebra, one finds that all first derivatives of f with respect to the probabilities cancel exactly, leaving finally the compact expression

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(\{p_R\})] = 4\gamma &\sum_{M,T,i,j,Q,S} F\left(\vec{r}_i^{\,R} - \vec{r}_j^{\,S}, t\right)\\ &\times m_i m_j \mathbb{E}\left[\frac{\partial^2 f(\{p_R\})}{\partial p_N \partial p_M} p_T p_M(\delta_{MQ} - p_Q)(\delta_{TS} - p_S)\right].\end{aligned} \tag{55}$$

Introducing the probability density $P(\{p_R\}, t)$, which includes as a factor the constraint $\delta\left(\sum_L p_L - 1\right)$ requiring that the probabilities sum to unity, we can also write the expectation of $(\mathrm{d}/\mathrm{d}t) f(\{p_R\})$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(\{p_R\})] = \prod_L \int \mathrm{d}p_L \frac{\partial P(\{p_R\})}{\partial t} f(\{p_R\}). \tag{56}$$

Comparison of this expression with (55), as rearranged by two integrations by parts (the surface terms when any probability is 0 or 1 do not contribute; see below), one gets the Fokker–Planck equation

$$\frac{\partial P(\{p_R\})}{\partial t} = \sum_{M,T}\frac{\partial^2}{\partial p_M \partial p_T}[A_{MT}(\{p_R\})P(\{p_R\}, t)], \tag{57}$$

with

$$\begin{aligned}A_{MT} &= 4\gamma p_M p_T \sum_{i,j,Q,S} F\left(\vec{r}_i^{\,R} - \vec{r}_j^{\,S}, t\right) m_i m_j(\delta_{MQ} - p_Q)(\delta_{TS} - p_S)\\ &= 4\gamma p_M p_T \sum_{i,j,Q,S} m_i m_j p_Q p_S\left[F\left(\vec{r}_i^{\,M} - \vec{r}_j^{\,T}, t\right) + F\left(\vec{r}_i^{\,Q} - \vec{r}_j^{\,S}, t\right)\right.\\ &\quad\left. - F\left(\vec{r}_i^{\,Q} - \vec{r}_j^{\,T}, t\right) - F\left(\vec{r}_i^{\,M} - \vec{r}_j^{\,S}, t\right)\right].\end{aligned} \tag{58}$$

This equation is a specific case of a general Fokker–Planck equation written by Pearle [5] as the basis for a general class of objective reduction models. When $F(\vec{x}-\vec{y},t)$ takes the factorized form of (25), A_{MT} can be rewritten as

$$A_{MT} = 4\gamma p_M p_T \sum_\alpha \mathcal{F}(\alpha,t)\phi_M\phi_T, \tag{59}$$

with

$$\phi_M = \sum_{i,Q} m_i p_Q\left[f\left(\alpha,\vec{r}_i^{\,M}\right) - f\left(\alpha,\vec{r}_j^{\,Q}\right)\right]. \tag{60}$$

We see that in addition to vanishing when either $p_M = 0$ or $p_T = 0$, A_{MT} vanishes when either $p_M = 1$ or $p_T = 1$, because ϕ_M vanishes when $p_M = 1$. This is why the integrations by parts leading to the Fokker–Planck equation produce no surface terms, and also why the Fokker–Planck equation of (57) satisfies the criteria that Pearle [5] formulated for getting a Fokker–Planck equation that leads to state vector reduction with Born rule probabilities.

As an application of (55), if we substitute $f(\{p_R\}) = p_L$ we find, since $\partial^2 p_L/(\partial p_N \partial p_M) = 0$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[p_L] = 0. \tag{61}$$

Similarly, if we substitute $f(\{p_R\}) = \delta_{KL}p_L - p_K p_L$, we find, using

$$\frac{\partial^2}{\partial p_M \partial p_N}[\delta_{KL}p_L - p_K p_L] = -[\delta_{MK}\delta_{NL} + \delta_{ML}\delta_{NK}], \tag{62}$$

that (55) yields (32). More generally, (55) and the corresponding Fokker–Planck equation of (57) allow one to calculate the time evolution of a general function $f(\{p_R\})$ of the probabilities.

5. Noise effects: energy production and radiation by atoms

The noise coupling postulated in section 1 as the origin of state vector reduction has other physical effects that serve to place upper bounds on the noise coupling strength γ. We focus in this section in particular on energy production, and gamma radiation from atoms, which place particularly stringent bounds on the model parameters.

5.1. Energy production

To calculate the mean rate of energy production, we have to evaluate $(\mathrm{d}/\mathrm{d}t)\mathrm{Tr}\, H\rho(t) = \mathrm{Tr}\, H(\mathrm{d}/\mathrm{d}t)\rho(t)$. From (8) we find, by repeated cyclic permutation under the trace, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, H\rho(t) = -\gamma \int \mathrm{d}^3x \int \mathrm{d}^3y \int_0^t \mathrm{d}s\, D(\vec{x}-\vec{y},t-s)\mathrm{Tr}([[H,M(\vec{x})],M(\vec{y},s-t)]\rho(t)). \tag{63}$$

This equation is exact through order γ. We now make the Markovian approximation, of assuming that we can ignore the 'memory effect' associated with the characteristic decay time of the noise correlator $D(\vec{x}-\vec{y},t-s)$, by replacing $M(\vec{y},s-t)$ by $M(\vec{y},0) = M(\vec{y})$. For white noise, where $D(\vec{x}-\vec{y},t-s) = G(\vec{x}-\vec{y})\delta(t-s)$, the Markovian approximation is exact; for non-white thermal noises, it should be a good approximation when the energy at the peak of the noise spectrum is much higher than the typical kinetic energies of the particles to which the noise couples (see appendix A). With this approximation, (63) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, H\rho(t) = -\gamma \int \mathrm{d}^3x \int \mathrm{d}^3y\, F(\vec{x}-\vec{y},t)\,\mathrm{Tr}([[H,M(\vec{x})],M(\vec{y})]\rho(t)), \tag{64}$$

where we have made use of definition (11).

Let us now assume that H is the nonrelativistic Hamiltonian for a collection of particles interacting through a general velocity-independent potential,

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2M_i} + V(\{\vec{q}_\ell\}), \tag{65}$$

while $M(\vec{x})$ has the form of (3) (in the mass-coupled noise case, $m_i = M_i$) and $F(\vec{x}-\vec{y},t)$ has the factor decomposition of (25). Then carrying out the $\vec{x}$ and $\vec{y}$ integrals, (64) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,H\rho(t) = -\gamma\sum_\alpha \mathcal{F}(\alpha,t)\sum_{i,j} m_i m_j\,\mathrm{Tr}([[H, f(\alpha,\vec{q}_i)], f(\alpha,\vec{q}_j)]\rho(t)). \tag{66}$$

The commutators appearing in (66) are easily evaluated,

$$\begin{aligned}[H, f(\alpha,\vec{q}_i)] &= \left[\sum_j \frac{\vec{p}_j^{\,2}}{2M_j} + V(\{\vec{q}_\ell\}), f(\alpha,\vec{q}_i)\right] \\ &= \left[\frac{\vec{p}_i^{\,2}}{2M_i}, f(\alpha,\vec{q}_i)\right] \\ &= \frac{-\mathrm{i}}{2M_i}\left[\vec{p}_i\cdot\vec{\nabla}_{\vec{q}_i} f(\alpha,\vec{q}_i) + \nabla_{\vec{q}_i} f(\alpha,\vec{q}_i)\cdot\vec{p}_i\right],\end{aligned} \tag{67}$$

giving

$$[[H, f(\alpha,\vec{q}_i)], f(\alpha,\vec{q}_j)] = -\delta_{ij}\frac{1}{M_i}[\vec{\nabla}_{\vec{q}_i} f(\alpha,\vec{q}_i)]^2. \tag{68}$$

Substituting this into (66), we obtain finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,H\rho(t) = \gamma\sum_\alpha \mathcal{F}(\alpha,t)\sum_i \frac{m_i^2}{M_i}\,\mathrm{Tr}([\vec{\nabla}_{\vec{q}_i} f(\alpha,\vec{q}_i)]^2\rho(t)). \tag{69}$$

A further simplification of this result can be achieved by using the Fourier transform representation of $F(\vec{x}-\vec{y},t)$, which (recalling the assumed spatial inversion invariance) takes the form

$$\begin{aligned}F(\vec{x}-\vec{y},t) &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot(\vec{x}-\vec{y}))\hat{F}(\vec{k},t) \\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}[\cos(\vec{k}\cdot\vec{x})\cos(\vec{k}\cdot\vec{y}) + \sin(\vec{k}\cdot\vec{x})\sin(\vec{k}\cdot\vec{y})]\hat{F}(\vec{k},t).\end{aligned} \tag{70}$$

This has the general structure of (25), with $\sum_\alpha$ corresponding to $\int \mathrm{d}^3k(2\pi)^{-3}\sum_{n=1}^{2}$, with n a discrete index distinguishing between the sine and cosine modes, that is, $\mathcal{F}(\alpha,t) = \hat{F}(\vec{k},t)$ for both $n = 0, 1$, and $f(\vec{k}, n = 0, \vec{x}) = \cos(\vec{k}\cdot\vec{x})$ and $f(\vec{k}, n = 1, \vec{x}) = \sin(\vec{k}\cdot\vec{x})$. Substituting (70) into (69) then gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,H\rho(t) &= \gamma\int\frac{\mathrm{d}^3k}{(2\pi)^3}\hat{F}(\vec{k},t)\sum_i\frac{m_i^2}{M_i}\,\mathrm{Tr}[\vec{k}^2(\cos^2(\vec{k}\cdot\vec{q}_i) + \sin^2(\vec{k}\cdot\vec{q}_i))\rho(t)] \\ &= \gamma\int\frac{\mathrm{d}^3k}{(2\pi)^3}\vec{k}^2\hat{F}(\vec{k},t)\sum_i\frac{m_i^2}{M_i},\end{aligned} \tag{71}$$

where in the final step we have used $\mathrm{Tr}\,\rho(t) = 1$. Thus, in the Markovian approximation, we get a simple formula for the energy production rate, expressed entirely in terms of the Fourier transform of $F(\vec{x}-\vec{y},t)$. We see that the dynamics of the density matrix $\rho(t)$ drops out of the final formula, as does the interaction potential in the Hamiltonian H, leaving a result that is just the sum of contributions from the kinetic terms of the individual particles.

We give now several specific examples of the formula (71). First of all, in the standard white noise CSL model, one has uncoupled space and time correlators of the product form

$$D(\vec{x}-\vec{y}, t-s) = G(\vec{x}-\vec{y})\delta(t-s), \tag{72}$$

which taking account of the fact that $\int_0^t \mathrm{d}s\,\delta(t-s) = 1/2$, gives

$$F(\vec{x}-\vec{y}, t) = \tfrac{1}{2}G(\vec{x}-\vec{y}). \tag{73}$$

The spatial correlation function $G(\vec{x}-\vec{y})$ is the autoconvolution of the function $g(\vec{x})$ introduced as the CSL smearing function,

$$G(\vec{x}-\vec{y}) = \int \mathrm{d}^3z\, g(\vec{x}-\vec{z})g(\vec{y}-\vec{z}), \qquad g(\vec{x}) = (\sqrt{2\pi}r_C)^{-3}\,\mathrm{e}^{-\vec{x}^2/(2r_C^2)}, \tag{74}$$

which, incidentally, gives an alternative form of factor decomposition for this model. We will continue, however, to use the factor decomposition given by the Fourier transform, which is

$$G(\vec{x}-\vec{y}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot(\vec{x}-\vec{y}))\,\mathrm{e}^{-\vec{k}^2r_C^2}, \tag{75}$$

so that substituting (75) and (73) into (70) and writing $k = |\vec{k}|$ we have

$$\hat{F}(\vec{k}, t) = \tfrac{1}{2}\,\mathrm{e}^{-k^2r_c^2}. \tag{76}$$

Substituting this into (71) gives for the white noise CSL model

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,H\rho(t) &= \frac{\gamma}{4\pi^2}\sum_i \frac{m_i^2}{M_i}\int_0^\infty \mathrm{d}k\,k^4\,\mathrm{e}^{-k^2r_C^2}\\ &= \frac{3\gamma}{32\pi^{3/2}r_C^5}\sum_i\frac{m_i^2}{M_i} = \frac{3\lambda}{4m_N^2r_C^2}\sum_i\frac{m_i^2}{M_i},\end{aligned} \tag{77}$$

with m_N the nucleon mass and $\lambda = \gamma m_N^2\big/\big(8\pi^{3/2}r_C^3\big)$ the alternative form of the noise coupling generally used in the CSL literature[7]. This result agrees with the standard answer for the constant energy production rate in the CSL model.

Consider next a variant of the product correlator model, in which there is a cutoff in the frequency spectrum, obtained by replacing $\gamma\delta(t-s)$ in (72) by

$$\delta_{\gamma(\omega)}(t-s) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\,\gamma(\omega)\cos(\omega(t-s)). \tag{78}$$

This replacement turns the original coupling γ into a frequency dependent coupling $\gamma(\omega)$, with the specialization back to constant γ given by $\delta_{\gamma(\omega)\equiv\gamma} = \gamma\delta(t-s)$. In this case we find that

$$\int_0^t \mathrm{d}s\,\delta_{\gamma(\omega)}(t-s) = \frac{1}{\pi}\int_0^\infty \gamma\left(\frac{u}{t}\right)\frac{\mathrm{d}u}{u}\sin u, \tag{79}$$

which approaches the constant $\gamma(0)/2$ as $t\to\infty$, with the entire contribution in the infinite time limit coming from the infrared region of the integral near $\omega = 0$. Thus even with a

[7] In the CSL model literature, what we here call γm_N^2 is termed γ, because the noise there is introduced as coupled to the nucleon number density rather than the mass density. Also, we note that the dimensionality of γ is determined by the dimensionality assigned to the field ϕ, and is not the same in our white noise and thermal model examples. In the white noise CSL model, what we call γ has dimensionality mass^{-4} in microscopic units with $\hbar = c = 1$, whereas in the thermal noise model discussed below, where ϕ is taken as a conventional dimension one boson field, γ has dimensionality mass^{-2} in microscopic units.

high-frequency cutoff, there is a constant energy production rate at large times in a model with uncoupled space and time correlators. To avoid getting a constant energy production rate in the product correlator model, one must include an infrared cutoff, by taking $\gamma(0) = 0$.

Finally, anticipating our discussion below of thermal noise, consider a correlator of the general form

$$D(\vec{x} - \vec{y}, t - s) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \omega_k} f(k) \cos(\vec{k} \cdot (\vec{x} - \vec{y})) \cos(\omega_k (t - s)), \tag{80}$$

with ω_k a wave number dependent angular frequency. Integrating to form $F(\vec{x} - \vec{y}, t)$, we have

$$F(\vec{x} - \vec{y}, t) = \int_0^t \mathrm{d}s\, D(\vec{x} - \vec{y}, t - s) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \omega_k^2} f(k) \cos(\vec{k} \cdot (\vec{x} - \vec{y})) \sin(\omega_k t), \tag{81}$$

which identifies the Fourier transform as

$$\hat{F}(\vec{k}, t) = \frac{f(k)}{\omega_k^2} \sin(\omega_k t). \tag{82}$$

Substituting this into (71) gives for the energy production rate

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}\, H\rho(t) = \gamma \sum_i \frac{m_i^2}{M_i} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{k^2 f(k)}{\omega_k^2} \sin(\omega_k t). \tag{83}$$

Even when $\omega_k \propto k$, this expression is strongly convergent in the infrared as a consequence of the vanishing of phase space for small k values. Hence if $f(k)$ is cut off sharply at large k values, as expected in thermal models, it leads to a vanishing energy production rate at large times by use of the Riemann–Lebesgue theorem. Integrating to find the total energy production $\Delta \mathrm{Tr}\, H\rho(t) \equiv \mathrm{Tr}\, H\rho(t) - \mathrm{Tr}\, H\rho(0)$, we find

$$\Delta \mathrm{Tr}\, H\rho(t) = \frac{\gamma}{2\pi^2} \sum_i \frac{m_i^2}{M_i} \int_0^\infty \mathrm{d}k \frac{k^4 f(k)}{\omega_k^3} [1 - \cos(\omega_k t)], \tag{84}$$

which as $t \to \infty$ gives, again by an application of the Riemann–Lebesgue theorem,

$$\Delta \mathrm{Tr}\, H\rho(\infty) = \frac{\gamma}{2\pi^2} \sum_i \frac{m_i^2}{M_i} \int_0^\infty \mathrm{d}k \frac{k^4 f(k)}{\omega_k^3}. \tag{85}$$

5.2. *Gamma radiation from atoms*

An important constraint on noise model parameters is provided by the spontaneous emission of gamma rays from atoms, a process first calculated for free electrons by Fu [6] and later calculated for general atomic systems by Adler and Ramazanoğlu [7]. Results were given in the latter paper for a correlator of the form $G(\vec{x} - \vec{y})\delta_{\gamma(\omega)}(t - s)$. Remembering that the CSL definition of γ is m_N^2 times the definition of γ used in this paper, and comparing (73) and (75) with (80), we see that the results of [7] can be converted to apply to a correlator of the form of (81) by the substitution

$$\frac{\gamma(\omega)}{m_N^2} \mathrm{e}^{-k^2 r_C^2} \to \gamma\pi \frac{f(k)}{\omega_k} \delta(\omega - \omega_k). \tag{86}$$

When ω_k has the form $\omega_k = \sqrt{k^2 + \mu^2}$, making this substitution into (44) of [7] gives as the formula for the power radiation $\mathrm{d}P$ per unit photon energy $\mathrm{d}p$ from a hydrogen atom,

$$\frac{\mathrm{d}P}{\mathrm{d}p} = 2\left[1 - \frac{1}{[1 + (pa_0/2)^2]^2}\right] \frac{\gamma e^2 (p^2 - \mu^2)^{3/2} f(\sqrt{p^2 - \mu^2})}{3\pi^2 p}, \tag{87}$$

with $e^2 \simeq 1/137.04$ and with $a_0 = 1/(e^2 m_{\mathrm{electron}}) = 0.529 \times 10^{-8}$cm the Bohr radius.

6. Models for the correlation function

We turn in this section to a discussion of specific models for the correlation function $D(\vec{x} - \vec{y}, t_1 - t_2)$ introduced in (5). We first briefly consider the standard CSL factorizable correlation function with white noise, and its variant with a cutoff in the noise spectrum, which has been the basis of most discussions to date of the phenomenology of objective reduction models. However, one would in general expect the spatial and temporal structures of the correlation function to be intertwined, and in particular, a correlation function arising from fields with a particle interpretation will have the spatial and temporal correlations coupled by a mass-shell constraint. This is the motivation for the models discussed in the remainder of this section, which are based on a classical model extracted from the quantum thermal Green's function of a boson of mass μ.

6.1. The product correlator model

The product model for the correlation function was written in (72) through (79). With a white noise spectrum, the standard CSL choice for the noise strength parameter is $m_N^2\gamma = 10^{-30}$ cm^3 s^{-1}, and the standard choice for the correlation length is $r_C \sim 10^{-5}$ cm. The white noise model with these parameter choices obeys all experimental upper bound constraints, and readily explains measurements in which $n_{\text{out}} \sim 10^{13}$ nucleons are displaced by a distance of at least r_C.

In [8], Adler gave a reanalysis of the upper and lower bounds on parameters in stochastic reduction models. Under the assumption that latent image formation, in either photography or etched track detectors, constitutes a measurement (rather than the measurement occurring only through the subsequent development that reveals the latent image), he concluded that the noise strength parameter γ should be larger than conventionally assumed in the CSL white noise model, by a factor of $2 \times 10^{9\pm2}$. This however conflicts with bounds set by Fu [6] and Adler and Ramazanoğlu [7] on spontaneous 11 keV gamma radiation emission from germanium, unless the white noise spectrum is cut off at energies below 11 keV by the spectral weight $\gamma(\omega)$ appearing in (78). Such a cutoff would still allow sufficiently rapid state vector reduction to account for observed measurement times, as already noted in the review of Bassi and Ghirardi [9]. Thus, the product model for the correlation function, with a high-frequency cutoff in the noise spectrum, is consistent both with all upper bounds, and with the assumption that latent image formation constitutes a measurement signaling state vector reduction. Such a correlation function might arise from a pre-quantum theory in which quantized fields are not the primary entities, as in [10]. But as already noted, a product correlation function is not expected to arise from quantum fields with a particle interpretation.

6.2. Thermal correlation function model

In this section, we shall motivate a model for the correlation function $D(\vec{x} - \vec{y}, t_1 - t_2)$ by considering the correlation function for a quantum field in a thermal state at temperature T. Let $\phi(\vec{x}, t)$ be a scalar quantum field, with the mode decomposition

$$\phi(\vec{x}, t) = \int d^3k \left[\frac{1}{2\omega_k(2\pi)^3}\right]^{1/2} [a(\vec{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + a^\dagger(\vec{k})\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}], \tag{88}$$

where $a(\vec{k})$ and $a^\dagger(\vec{k})$ are the mode annihilation and creation operators, and where the mode energy ω_k is

$$\omega_k = \sqrt{\vec{k}^2 + \mu^2}, \tag{89}$$

with μ the scalar field mass. We have, as before, set Planck's constant $\hbar$ equal to unity, and also set the Boltzmann constant equal to unity, so that in a thermal state at temperature T the expectations of products of creation and annihilation operators are given by

$$\langle a(\vec{k})a^\dagger(\vec{k}')\rangle = \delta^3(\vec{k}-\vec{k}')[1+N(\vec{k})], \qquad \langle a^\dagger(\vec{k})a(\vec{k}')\rangle = \delta^3(\vec{k}-\vec{k}')N(\vec{k}), \tag{90}$$

with the mean occupation number $N(\vec{k})$ given by

$$N(\vec{k}) = \frac{1}{\mathrm{e}^{\frac{\omega_k}{T}}-1}. \tag{91}$$

From these equations we can now calculate the correlation function

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle = \int \mathrm{d}^3k \frac{1}{2\omega_k(2\pi)^3}[[1+N(\vec{k})]\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y})-\omega_k(t_1-t_2))} + N(\vec{k})\,\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y})-\omega_k(t_1-t_2))}], \tag{92}$$

which can be written as the sum of a temperature-independent part $\Delta_+(\vec{x}-\vec{y},t_1-t_2)$ and a temperature-dependent part $D(\vec{x}-\vec{y},t_1-t_2)$ as follows:

$$\begin{aligned}
\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle &= \Delta_+(\vec{x}-\vec{y},t_1-t_2) + D(\vec{x}-\vec{y},t_1-t_2),\\
\Delta_+(\vec{x}-\vec{y},t_1-t_2) &= \int \mathrm{d}^3k \frac{1}{2\omega_k(2\pi)^3}\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y})-\omega_k(t_1-t_2))},\\
D(\vec{x}-\vec{y},t_1-t_2) &= \int \mathrm{d}^3k \frac{N(\vec{k})}{2\omega_k(2\pi)^3}[\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y})-\omega_k(t_1-t_2))} + \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y})-\omega_k(t_1-t_2))}].
\end{aligned} \tag{93}$$

In the zero temperature limit, $D(\vec{x},t)$ vanishes, and (93) reduces to the temperature-independent piece Δ_+, which is one of the standard relativistic quantum theory vacuum Green's functions arising directly from the non-commutativity of $a(\vec{k})$ and $a^\dagger(\vec{k})$, and is a complex number for general arguments. The real-valued temperature-dependent piece $D(\vec{x}-\vec{y},t_1-t_2)$, on the other hand, is invariant under the interchange $\vec{x},t_1 \leftrightarrow \vec{y},t_2$, and therefore can serve as a model for the expectation of real, classical, commuting noise fields introduced in (5).

Since $N(\vec{k})$ and ω_k are even in $\vec{k}$, writing $\mathrm{e}^{\pm\mathrm{i}(\vec{k}\cdot(\vec{x}-\vec{y}))} = \cos(\vec{k}\cdot(\vec{x}-\vec{y})) \pm \mathrm{i}\sin(\vec{k}\cdot(\vec{x}-\vec{y}))$, the sine functions average to zero, and the formula for $D(\vec{x}-\vec{y},t_1-t_2)$ simplifies to

$$D(\vec{x}-\vec{y},t_1-t_2) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{N(\vec{k})}{\omega_k}\cos(\vec{k}\cdot(\vec{x}-\vec{y}))\cos(\omega_k(t_1-t_2)), \tag{94}$$

which has the form assumed in (80), with $f(k) = N(\vec{k})$ as given in (91) and with ω_k the energy–momentum relation given in (89). We shall slightly generalize the model specified by (94) and (91), by introducing a thermodynamic chemical potential ζ into the occupation number, which we thus write as

$$N(\vec{k}) = \frac{1}{\mathrm{e}^{\frac{\omega_k-\zeta}{T}}-1}, \tag{95}$$

which allows us to accommodate systems with general particle density [11]. For the case of noise fields associated with particles having a standard energy–momentum dispersion relation, (94), (89) and (95) constitute our basic model for the correlation function $D(\vec{x}-\vec{y},t_1-t_2)$. Corresponding to this model, the function $F(\vec{x}-\vec{y},t)$ defined in (11) is given by

$$F(\vec{x}-\vec{y},t) = \int_0^t \mathrm{d}s\, D(\vec{x}-\vec{y},t-s) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{N(\vec{k})}{\omega_k^2}\cos(\vec{k}\cdot(\vec{x}-\vec{y}))\sin(\omega_k t), \tag{96}$$

and the integral appearing in the rate function $\Gamma(t)$ of (17) is given by

$$I(\vec{x}-\vec{y},t) \equiv \int_0^t \mathrm{d}s\, F(\vec{x}-\vec{y},s) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{N(\vec{k})}{\omega_k^3}\cos(\vec{k}\cdot(\vec{x}-\vec{y}))[1-\cos(\omega_k t)]. \tag{97}$$

6.3. Dilute and nonrelativistic limits

Let us consider now the dilute limit of (94) and (95), obtained [11] by letting the chemical potential ζ be large and negative, so that $N(\vec{k})$ becomes

$$N(\vec{k}) \simeq \mathrm{e}^{-\frac{\omega_k-\zeta}{T}}. \tag{98}$$

We will be particularly interested in applying (98) to the nonrelativistic case $T \ll \mu$, where we can expand

$$\omega_k = \sqrt{\vec{k}^2+\mu^2} \simeq \mu + \frac{\vec{k}^2}{2\mu}, \tag{99}$$

so that $N(\vec{k})$ becomes

$$N(\vec{k}) = \mathrm{e}^{-(\mu-\zeta)/T}\,\mathrm{e}^{-\vec{k}^2/(2\mu T)}. \tag{100}$$

Where ω_k appears as a denominator factor in (94), (96), and (97), it can be approximated by μ, so these equations become respectively

$$\begin{aligned}
D(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\vec{k}^2/(2\mu T)}\cos(\vec{k}\cdot\vec{x})\cos\left(\left(\mu+\frac{\vec{k}^2}{2\mu}\right)t\right),\\
F(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\vec{k}^2/(2\mu T)}\cos(\vec{k}\cdot\vec{x})\sin\left(\left(\mu+\frac{\vec{k}^2}{2\mu}\right)t\right),\\
I(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^3}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\vec{k}^2/(2\mu T)}\cos(\vec{k}\cdot\vec{x})\left[1-\cos\left(\left(\mu+\frac{\vec{k}^2}{2\mu}\right)t\right)\right].
\end{aligned} \tag{101}$$

Carrying out the angular averaging over $\vec{k}$, remembering that it is the difference $D(\vec{0},t) - D(\vec{x},t)$ that enters into the reduction formalism, and writing $R = |\vec{x}|$, $k = |\vec{k}|$, (101) yields

$$\begin{aligned}
D(\vec{0},t) - D(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu}\int_{-\infty}^{\infty}\frac{k^2\mathrm{d}k}{(2\pi)^2}\,\mathrm{e}^{-k^2/(2\mu T)}\left[1-\frac{\sin(kR)}{kR}\right]\cos\left(\left(\mu+\frac{k^2}{2\mu}\right)t\right),\\
F(\vec{0},t) - F(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^2}\int_{-\infty}^{\infty}\frac{k^2\mathrm{d}k}{(2\pi)^2}\,\mathrm{e}^{-k^2/(2\mu T)}\left[1-\frac{\sin(kR)}{kR}\right]\sin\left(\left(\mu+\frac{k^2}{2\mu}\right)t\right),\\
I(\vec{0},t) - I(\vec{x},t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^3}\int_{-\infty}^{\infty}\frac{k^2\mathrm{d}k}{(2\pi)^2}\,\mathrm{e}^{-k^2/(2\mu T)}\left[1-\frac{\sin(kR)}{kR}\right]\\
&\quad\times\left[1-\cos\left(\left(\mu+\frac{k^2}{2\mu}\right)t\right)\right].
\end{aligned} \tag{102}$$

The integrals in (102) can all be evaluated from the formula

$$\int_{-\infty}^{\infty} x^2\,\mathrm{d}x\,\exp(-\alpha x^2)\frac{\sin(x\beta)}{x\beta} = \frac{\sqrt{\pi}}{2\alpha^{3/2}}\,\mathrm{e}^{-\beta^2/(4\alpha)}, \tag{103}$$

with results that are summarized in appendix B. In particular, for large times, the formula for $I(\vec{0},t) - I(\vec{x},t)$ limits to

$$I(\vec{0},t=\infty) - I(\vec{x},t=\infty) \simeq \mathrm{e}^{-(\mu-\zeta)/T}\left(\frac{T}{2\pi\mu}\right)^{3/2}[1-\mathrm{e}^{-R^2\mu T/2}] \tag{104}$$

and the formulae of appendix B show that the characteristic reduction time t_R for approach to the asymptotic value of (104) is the inverse temperature T^{-1}.

To compare this to the standard CSL model formulae, let us look at the decay of the off-diagonal density matrix element of a one-particle system of mass equal to the nucleon mass

m_N, which we have seen is governed by the same rate function Γ as state vector reduction. From (16) and (19), at large times one has

$$\langle \vec{r}^{\,1}|\rho(t=\infty)|\vec{r}^{\,2}\rangle = \mathrm{e}^{-\Gamma(\infty)}\langle \vec{r}^{\,1}|\rho(0)|\vec{r}^{\,2}\rangle, \tag{105}$$

with

$$\begin{aligned}\Gamma(t=\infty) &= 2\gamma m_N^2[I(\vec{0},\infty) - I(\vec{r}^{\,1}-\vec{r}^{\,2},\infty)] \\ &= 2\gamma m_N^2\,\mathrm{e}^{-(\mu-\zeta)/T}\left(\frac{T}{2\pi\mu}\right)^{3/2}[1-\mathrm{e}^{-R^2\mu T/2}],\end{aligned} \tag{106}$$

where we have written $R = |\vec{r}^{\,1} - \vec{r}^{\,2}|$. The comparable formula in the CSL model is given in (8.15) of [9],

$$\langle \vec{r}^{\,1}|\rho(t)|\vec{r}^{\,2}\rangle = \mathrm{e}^{-\Gamma^{\mathrm{CSL}}(t)}\langle \vec{r}^{\,1}|\rho(0)|\vec{r}^{\,2}\rangle, \tag{107}$$

where $\Gamma^{\mathrm{CSL}}(t)$ is given by

$$\Gamma^{\mathrm{CSL}}(t) = t\gamma^{\mathrm{CSL}}\left(\frac{1}{4\pi r_c^2}\right)^{3/2}[1-\mathrm{e}^{-R^2/(4r_c^2)}], \tag{108}$$

and where (as remarked above in a footnote) γ^{CSL} is what we call γm_N^2. We see that the functional form of the R-dependence in (104) and (108) is the same, with the CSL model correlation length r_C related to the nonrelativistic thermal model parameters by

$$r_C^2 = \frac{1}{2\mu T}, \qquad \left(\frac{T}{2\pi\mu}\right)^{3/2} = \mu^{-3}\left(\frac{1}{4\pi r_C^2}\right)^{3/2}. \tag{109}$$

However, whereas $\Gamma^{\mathrm{CSL}}(t)$ grows linearly with time for large times t, in the thermal noise model $\Gamma(t=\infty)$ approaches a constant. This means that to achieve the degree of density matrix diagonalization, or state vector reduction, attained in the CSL model in time Δt, the parameters in the thermal model must obey

$$\Delta t\gamma^{\mathrm{CSL}} = \frac{2\gamma m_N^2}{\mu^3}\mathrm{e}^{-(\mu-\zeta)/T}. \tag{110}$$

6.4. Can thermalized dark matter be the noise source?

As we have already noted, one motivation for studying non-white noise is to investigate whether there can be a cosmological origin for the noise that drives state vector reduction in objective reduction models. Since there is now strong evidence that about a quarter of the closure density of the universe consists of dark matter, and since weakly interacting massive particle (WIMP) candidates for dark matter are expected to be thermalized, it is natural to apply the results of the preceding section to an analysis of whether dark matter can account for the noise coupling in (6). We will not attempt to discuss here the necessary conditions for dark matter to give a real-valued, as opposed to an imaginary-valued, noise term in the Schrödinger equation; this important question will be deferred to future work. What we shall do in this section is to assume that a real-valued noise coupling can be achieved, and to investigate the phenomenological implications of assuming that state vector reduction is associated with observed dark matter parameters.

A few basic facts about dark matter are needed. If dark matter is due to WIMPs, then observational evidence [12] suggests a WIMP distribution in the galactic halo of mass density $\rho_{\mathrm{mass}} = 0.3$ GeV cm^{-3}, and a Maxwellian velocity distribution with $v_{\mathrm{rms}} = 220$ km s^{-1} = $7.3\times10^{-4}c$. The rms velocity is estimated from the formula

$$\mu v^2/r_{\mathrm{galaxy}} = GM_{\mathrm{galaxy}}\mu\big/r_{\mathrm{galaxy}}^2, \tag{111}$$

which describes the gravitational binding of WIMPs of mass μ to the galaxy of mass M_{galaxy}, at radius r_{galaxy}, with G the Newton gravitational constant. Direct limits on possible solar system-bound dark matter are weaker [13, 14] by a factor of 3×10^5, that is, $\rho_{\text{mass ss}} \leqslant 0.9 \times 10^5$ GeV cm^{-3}. There is at present no observational limit on possible earth-bound dark matter. If there were solar system-bound dark matter, around the radius of the earth's orbit the rms velocity, by (111) would be $v_{\text{rms}} \sim 30$ km s^{-1} $= 10^{-4}c$, and for earth-bound dark matter, at the radius of the earth's surface, the rms velocity would be $v_{\text{rms}} \sim 8\,\text{km}\,\text{s}^{-1} = 0.27 \times 10^{-4}c$.

Because the WIMP mass μ cancels out of (111), there is currently no direct information about the dark matter particle mass. Dark matter particles coupling to the mass density cannot be too light, or they would conflict with gravitational fifth force experiments. If we write the noise coupling as

$$\gamma = \frac{1}{M^2}, \tag{112}$$

then the fifth force experiments require

$$\frac{\exp(-\mu/\mu_5)}{M^2} \leqslant \frac{1}{M_{\text{Planck}}^2}, \tag{113}$$

with μ_5 the fifth force scale limit, currently [15] around $\mu_5 \sim 1.4 \times 10^{-3}$ eV. This gives the lower bound on M,

$$M \geqslant 10^{19-0.22\mu/\mu_5}\ \text{GeV}. \tag{114}$$

In addition to this constraint, there are also model-dependent astrophysical limits on the dark matter mass; for example, warm dark matter candidates must have masses greater than 1 keV [16].

For a Maxwellian distribution with $N(\vec{k})$ given by (100), the rms velocity is given by

$$v_{\text{rms}}^2 = \frac{3T}{\mu}, \tag{115}$$

so that using (109) we have

$$v_{\text{rms}} = \frac{\sqrt{3/2}}{\mu r_C}. \tag{116}$$

Hence for a given dark matter rms velocity, the correlation length r_C and the dark matter temperature T are determined as functions of the dark matter mass μ,

$$r_C = \frac{\sqrt{3/2}}{\mu v_{\text{rms}}}, \qquad T = \frac{\mu v_{\text{rms}}^2}{3}. \tag{117}$$

Integrating $N(\vec{k})$ over phase space, the number density ρ_n is given by

$$\begin{aligned} \rho_n \equiv \rho_m/\mu &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} N(\vec{k}) \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{-(\mu-\zeta)/T}\, \mathrm{e}^{-\vec{k}^2 r_C^2} = \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{8\pi^{3/2} r_C^3}, \end{aligned} \tag{118}$$

which determines the factor containing the chemical potential ζ in terms of ρ_m, μ and r_C,

$$\mathrm{e}^{-(\mu-\zeta)/T} = \frac{\rho_m}{\mu} 8\pi^{3/2} r_C^3. \tag{119}$$

From these equations, together with (106) and (109), and the assumption that the lower bound of (38) gives a good approximation to the reduction factor[8], we get the following estimate:

$$\text{Reduction factor} \sim \mathrm{e}^{-2\Gamma(t=\infty)}, \qquad 2\Gamma(t=\infty) = 4\left(\frac{m_N}{M}\right)^2 \frac{\rho_m}{\mu^4} n^2 N. \tag{120}$$

[8] Note, however, that although $\Gamma(t = \infty)$ is positive, the uniform positivity assumption on the integrand used to derive the upper and lower bounds is not obeyed in the thermal model; see the formulae in appendix B. Also, the simple model analyzed in appendix D gives exponential reduction as in the lower bound, but with a reduction factor $\mathrm{e}^{-\Gamma(t=\infty)}$, and so the rates calculated from the lower bound may be optimistic by a factor of 2.

Table 1. Correlation length r_C (cm) versus μ (keV) and rms velocity v.

$\mu \rightarrow$	1	10	10^2	10^3	10^4	10^6
v_h	3×10^{-5}	3×10^{-6}	3×10^{-7}	3×10^{-8}	3×10^{-9}	3×10^{-11}
v_s	2×10^{-4}	2×10^{-5}	2×10^{-6}	2×10^{-7}	2×10^{-8}	2×10^{-10}
v_e	9×10^{-4}	9×10^{-5}	9×10^{-6}	9×10^{-7}	9×10^{-8}	9×10^{-10}

Table 2. Reduction time t_R (s) versus μ (keV) and rms velocity v.

$\mu \rightarrow$	1	10	10^2	10^3	10^4	10^6
v_h	4×10^{-12}	4×10^{-13}	4×10^{-14}	4×10^{-15}	4×10^{-16}	4×10^{-18}
v_s	2×10^{-10}	2×10^{-11}	2×10^{-12}	2×10^{-13}	2×10^{-14}	2×10^{-16}
v_e	3×10^{-9}	3×10^{-10}	3×10^{-11}	3×10^{-12}	3×10^{-13}	3×10^{-15}

Here, in accordance with the properties of Γ discussed in section 2, n is the number of displaced nucleons that are bunched within a correlation length r_C and N is the number of such bunches of displaced nucleons.

Using (117) and (120), we can now make some estimates of the effectiveness of thermal dark matter in producing state vector reduction in the mass–density coupled model. Rewriting (117) in the form

$$r_C = \frac{0.24 \times 10^{-13}}{v_{\rm rms}} \frac{1\ {\rm GeV}}{\mu}\ {\rm cm}, \qquad T^{-1} = t_R = \frac{0.2 \times 10^{-23}}{v_{\rm rms}^2} \frac{1\ {\rm GeV}}{\mu}\ {\rm s}, \tag{121}$$

we get the following tables of values. For the correlation length r_C in the body of the table in cm, versus the dark matter mass μ in keV and its rms velocity appropriate to the galactic halo ($v_h = 220$ km s^{-1}), solar system-bound dark matter ($v_s = 30$ km s^{-1}) and earth-bound dark matter ($v_e = 8$ km s^{-1}), we have table 1.

Similarly, for the reduction time t_R in seconds in the body of the table versus the dark matter mass and its rms velocity, we have table 2.

Solving (120) for the value of $\gamma\rho_m$ which yields $2\Gamma(t = \infty) = 1$, which is the minimum value of the exponent beyond which reduction of the state vector starts to occur, we get

$$\gamma\rho_m = \frac{1.5 \times 10^{13}}{n^2 N} \left(\frac{\mu}{1{\rm GeV}}\right)^2 {\rm GeV\ cm}^{-1}. \tag{122}$$

From this, we get further tables of values. For $\gamma\rho_m$ in the body of the table, in GeVcm^{-1}, versus the dark matter mass μ in keV, and the effective number of displaced nucleons $n_{\rm out} = n^2 N = 10^{22}$ corresponding [9] to the standard CSL model, or $n_{\rm out} = n^2 N = 10^8$ corresponding to estimates [8] based[9] on latent image formation, we have table 3.

If we make the assumption that $\gamma = 1({\rm TeV})^{-2} = 10^{-6}({\rm GeV})^{-1}$, we get a table of values giving ρ_m in the body of the table, in GeVcm^{-3}, versus the dark matter mass and the effective number $n_{\rm out}$ of displaced nucleons (table 4).

From these tables, we see that state vector reduction, by the standard CSL criterion ($n_{\rm out} = 10^{22}$), and with a correlation length within a decade of the standard CSL value

[9] In the CSL model, one assumes $n = 10^9$, which is the number of nucleons in a volume of linear dimension 10^{-5}cm, and $N = 10^4$, giving $n^2 N = 10^{22}$. The latent image estimates of [8] take $n = 5640$ and $N = 20$, giving $n^2 N \sim 10^8$. The CSL model assumes a reduction rate of 10^7s^{-1}, whereas the latent image estimates assume a much smaller reduction rate of 30 s^{-1}, which is why in a white noise model the ratio of the noise strengths between the two cases is $\sim 10^9$, rather than the ratio $\sim 10^{14}$ of the $n^2 N$ values.

Table 3. $\gamma\rho_m$ (GeV cm^{-1}) versus μ (keV) and n_{out}.

$\mu \rightarrow$	1	10	10^2	10^3	10^4	10^6
10^{22}	2×10^{-21}	2×10^{-19}	2×10^{-17}	2×10^{-15}	2×10^{-13}	2×10^{-9}
10^8	2×10^{-7}	2×10^{-5}	2×10^{-3}	2×10^{-1}	2×10	2×10^5

Table 4. ρ_m (GeV cm^{-3}) versus μ (keV) and n_{out}.

$\mu \rightarrow$	1	10	10^2	10^3	10^4	10^6
10^{22}	3	3×10^4	3×10^8	3×10^{12}	3×10^{16}	3×10^{24}
10^8	3×10^{14}	3×10^{18}	3×10^{22}	3×10^{26}	3×10^{30}	3×10^{38}

$r_C = 10^{-5}$cm, is achievable in the dark matter model for dark matter masses in the range of 1–10 keV, with $\gamma \sim 1$ TeV^{-2} and with ρ_m below the current upper limit on solar system-bound dark matter. Adopting the latent image criterion ($n_{\text{out}} = 10^8$) requires dark matter densities that are much too large, so either the latent image analysis of [8] needs modification, or the dark matter model is unworkable.

For a dark matter mass μ of a kilovolt or greater, and the current limit on the fifth force scale μ_5, the fifth force bound of (113) becomes

$$M \geqslant 10^{19-0.15\times 10^6}, \tag{123}$$

which is strongly obeyed for the M values in the GeV to TeV range that are interesting. Referring to the discussion following (94), and using (95) and (117), we see that the function $f(\sqrt{p^2-\mu^2})$ in the formula (87) for the radiated gamma power from a hydrogen atom becomes

$$f(\sqrt{p^2-\mu^2}) = \frac{1}{\mathrm{e}^{(p-\zeta)/T}-1} \simeq e^{-(\mu-\zeta)/T}\,\mathrm{e}^{-(p-\mu)/T} = \frac{\rho_m}{\mu} 8\pi^{3/2} r_C^3\, \mathrm{e}^{-3(p-\mu)/(\mu v_{\text{rms}}^2)}. \tag{124}$$

Since for μ in the 1–10 keV range and for $p = 11$ keV, we have

$$\frac{3(p-\mu)}{\mu v_{\text{rms}}^2} \geqslant 6 \times 10^5, \tag{125}$$

the negative exponential in the final factor of (124) dominates all other factors in this equation and in (87), and so the experimental bound on 11 keV gamma radiation is strongly satisfied.

For both values of n_{out} displayed in the tables, the reduction time is sufficiently rapid, shorter than a few times 10^{-9} s, to account for realizable measurements. Finally, the total energy imparted by the noise to an isolated nucleon is obtained by evaluating (85) by using the form for $f(k)$ in the dilute nonrelativistic thermal model, giving

$$\mathrm{Tr}\, H\rho(t=\infty) = \frac{3m_N\gamma\rho_m}{2r_C^2\mu^4}. \tag{126}$$

For the CSL value of n_{out}, this is smaller than 10^{-15} K for all values of the dark matter velocity and mass in the tables, and so is sufficiently small so as to be unobservable.

The conclusion from this analysis is that, *if* dark matter couplings to ordinary matter have the anti-self-adjoint component needed to give a real-valued noise term in the Schrödinger equation, and *if* dark matter densities in the vicinity of earth are larger than the galactic halo density, but within current limits on solar system-bound dark matter, one could realize the standard CSL reduction model with the standard parameter values, and obey various important experimental constraints. The italicized assumptions make this mechanism for realizing state vector reduction conjectural; at worst, we have given an interesting toy model for reduction incorporating a non-white noise with a mass-shell constraint.

6.5. Thermal unparticles as the noise source

Recently Georgi [23] has introduced the concept of what he terms an 'unparticle', a field characterizing a scale-invariant sector of a low-energy effective field theory. This is of interest for collapse models, since if the noise field of (6) is the low-energy manifestation of a pre-quantum dynamics, such as discussed in the book [10], it is plausible that it could have a scale-invariant structure. Moreover, such an unparticle field, if a cosmological relic field, will have a thermal correlation structure. The concept of thermal unparticles has been introduced in a recent paper of Chen *et al* [24], who construct the thermal unparticle partition function by using the observation of Krasnikov [25], that an unparticle field can be constructed as a field with a continuous distribution of mass μ^2, characterized by a scale-invariant spectral function $\rho(\mu^2) \propto (\mu^2)^{d-2}$. More specifically, one obtains the unparticle propagator and partition function by integrating the corresponding propagator and partition function for a scalar field of squared mass μ^2 over the range $0 \leqslant \mu^2 \leqslant \infty$, with weighting function $\rho(\mu^2) = (d-1)\Lambda^{2(1-d)}(\mu^2)^{d-2}$. Here d is the anomalous scaling dimension characterizing unparticle physics, and Λ is a scale parameter (the cutoff for the low-energy effective theory) with dimension of mass[10].

In appendix E, we use the same method to construct the unparticle thermal correlation function from the thermal correlation function of (94) and (95) for a scalar field of mass μ^2. From this correlation function, we calculate the integrals needed to study both the state vector reduction rate and the noise-induced energy production. We recapitulate here two key formulae obtained from appendix E, both of which apply to a one-particle system of mass m. For the decay rate $\Gamma(t)$ of the off-diagonal matrix element $\langle\vec{x}|\rho(t)|\vec{0}\rangle$, which we have seen is also the reduction rate, we have

$$\begin{aligned}\Gamma(t) &= 2\gamma m^2[I_{\mathcal{U}}(\vec{0},t) - I_{\mathcal{U}}(\vec{x},t)] \\ &= \frac{\gamma m^2\Lambda^{2(1-d)}}{\pi^2}\int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-3}[1-\cos(\omega t)]}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1} \\ &\quad\times \int_0^1 \mathrm{d}v[1-\cos(v\omega|\vec{x}|)](1-v^2)^{d-1},\end{aligned} \tag{127}$$

where the subscript $\mathcal{U}$ on I corresponds to the notation of (E.16) of appendix E. For the noise-induced energy acquisition rate and total energy acquired by a particle of mass m, we have from (E.20) and (E.21) of appendix E,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,H\rho(t) = \frac{3\gamma m\Lambda^{2(1-d)}}{(2\pi)^2}\frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)}\int_0^\infty \mathrm{d}\omega \frac{\omega^{2d}\sin(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}, \tag{128}$$

and

$$\mathrm{Tr}\,H\rho(t) - \mathrm{Tr}\,H\rho(0) = \frac{3\gamma m\Lambda^{2(1-d)}}{(2\pi)^2}\frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)}\int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-1}[1-\cos(\omega t)]}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}. \tag{129}$$

Turning our attention first to (127), we note that the inner integral over v is always convergent at $v = 0$, and is convergent at $v = 1$ for Re $d > 0$. Because the inner integral

[10] Strictly speaking, the integration over μ^2 should extend only up to Λ^2, but when the temperature $T \ll \Lambda$, the integration for the partition function and thermal correlation function is effectively cut off by $N(\vec{k})$ of (95), and so negligible error is made in extending the upper limit to ∞.

in (127) vanishes as ω^2 for small ω, the integral over ω in (127) has precisely the same convergence properties at $\omega = 0$ as the integral giving the total energy production in (129). To study convergence, there are two cases to consider, (i) the chemical potential ζ is negative and nonzero and (ii) the chemical potential ζ is zero[11].

In the first case, of strictly negative ζ, the denominator $e^{\frac{\omega-\zeta}{T}} - 1$ is nonzero even at $\omega = 0$, and the integrals of (127) and (129) converge at $\omega = 0$ even when the factor $1 - \cos(\omega t)$ is replaced by unity, as long as $\mathrm{Re}\, d > 0$. So in this case we can extract the infinite time limit by invoking the Riemann–Lebesgue theorem, and simply dropping the term $\cos(\omega t)$ in (127) and (129), giving the formulae

$$\Gamma(\infty) = \frac{\gamma m^2 \Lambda^{2(1-d)}}{\pi^2} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-3}}{e^{\frac{\omega-\zeta}{T}} - 1} \int_0^1 \mathrm{d}v [1 - \cos(v\omega|\vec{x}|)](1 - v^2)^{d-1} \tag{130}$$

and

$$\mathrm{Tr}\, H\rho(\infty) - \mathrm{Tr}\, H\rho(0) = \frac{3\gamma m \Lambda^{2(1-d)}}{(2\pi)^2} \frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-1}}{e^{\frac{\omega-\zeta}{T}} - 1}. \tag{131}$$

Corresponding to the fact that the total energy production is finite, the energy production rate of (128) vanishes at large time. Referring now to (130), we see that there are two subcases governing the large $|\vec{x}|$ behavior, which we call (ia) and (ib). In subcase (ia), corresponding to $\mathrm{Re}\, d > 1$, the ω integral is convergent without using the ω^2 factor arising from the inner integral. So in this subcase we can apply the Riemann–Lebesgue theorem to the inner integral in the limit of large $|\vec{x}|$, by dropping the term $\cos(v\omega|\vec{x}|)$, leading to the conclusion that $\Gamma(\infty)$ varies from 0 at $|\vec{x}| = 0$ to a finite value at $|\vec{x}| = \infty$. In subcase (ib), corresponding to $1 \geqslant \mathrm{Re}\, d > 0$, the ω^2 factor from the inner integral is needed for convergence, and on changing integration variable from ω to $u = \omega|\vec{x}|$ one sees that $\Gamma(\infty)$ grows as $|\vec{x}|^{2(1-d)}$ as $|\vec{x}| \to \infty$.

In the second case, of vanishing chemical potential ζ, the denominator $e^{\frac{\omega-\zeta}{T}} - 1$ vanishes at $\omega = 0$, and the integrals of (127) and (129) now behave for small ω as

$$\Gamma(t) \sim \frac{\gamma m^2 T \Lambda^{2(1-d)}}{\pi^2} \int_0 \mathrm{d}\omega\, \omega^{2d-4} [1 - \cos(\omega t)] \int_0^1 \mathrm{d}v [1 - \cos(v\omega|\vec{x}|)](1 - v^2)^{d-1} \tag{132}$$

and

$$\mathrm{Tr}\, H\rho(t) - \mathrm{Tr}\, H\rho(0) \sim \frac{3\gamma m T \Lambda^{2(1-d)}}{(2\pi)^2} \frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)} \int_0 \mathrm{d}\omega\, \omega^{2d-2} [1 - \cos(\omega t)]. \tag{133}$$

There are now two subcases, which we label (iia) and (iib). In subcase (iia) we have $d > 1/2$, and both integrals (132) and (133) converge at $\omega = 0$ without using the ω^2 factor that comes from $1 - \cos(\omega t)$. So in this case, which behaves much like case (i), we can apply the Riemann–Lebesgue theorem to take the limit as $t \to \infty$ by dropping the term $\cos(\omega t)$, leading to finite values for $\Gamma(\infty)$ and $\mathrm{Tr}\, H\rho(\infty) - \mathrm{Tr}\, H\rho(0)$. One can then proceed to analyze the large $|\vec{x}|$ behavior of $\Gamma(\infty)$, as was done previously in case (i), with the conclusion that this is finite for $d > 3/2$ and it behaves as $|\vec{x}|^{3-2d}$ for $3/2 \geqslant d > 0$. In subcase (iib), we have $1/2 \geqslant d > 0$, and the ω^2 coming from the factor $1 - \cos(\omega t)$ is needed for convergence; defining a new integration variable $u = \omega t$, we see that both $\Gamma(t)$ and $\mathrm{Tr}\, H\rho(t)$ grow as t^{1-2d} in the large t limit, and correspondingly, the energy production rate decreases as t^{-2d}. So for vanishing chemical potential, and $1/2 > d > 0$, we have the interesting situation that one achieves perfect reduction at infinite time (that is, $\Gamma(\infty) = \infty$), although the reduction rate and the

[11] The chemical potential must always be less than or equal to zero, so there is not a third case of positive ζ, which would correspond to a physical region pole in the integrands coming from the vanishing of the denominator $e^{\frac{\omega-\zeta}{T}} - 1$ in all three integrals.

total energy production both grow as a fractional power of t, rather than linearly with t as in the standard CSL model. Correspondingly, the energy production rate vanishes as a fractional power of t at large time, which should make it easy to satisfy cosmological constraints [8] on the noise strength parameter.

We conclude that the thermal unparticle model exhibits a range of interesting behaviors, depending on the values of the chemical potential ζ and the unparticle dimension d. In addition to these two parameters, the effective noise strength $\gamma\Lambda^{2(1-d)}$ and the temperature T are also parameters of the model. Given the complexities of this four-dimensional parameter space, we do not attempt phenomenological fits of the model to experimental constraints on the noise strength, but this is clearly an interesting topic for future investigation.

7. Summary and discussion

We now summarize what has been done in the preceding sections and what is in the appendices, and sketch some directions for extensions of our investigations. In sections 1–5 we have continued the study of non-white noise models initiated in (I), focusing on the special case in which the noise field couples to the particle density. The analyses of sections 2, 3 and appendix D identify the characteristic rate functions governing density matrix diagonalization and state vector reduction, and show that both processes are exponential with the same rate function, in the simplified case (a single particle in a superposition of two localized states) discussed in appendix D. In section 4, we completed our formal analysis for non-white noise by deriving the corresponding Fokker–Planck equation, allowing us to make contact with earlier work of Pearle [5]. In section 5, with an eye toward phenomenological applications, we analyzed energy production and gamma radiation by atoms in terms of the correlation functions of the non-white noise model.

In section 6, we turned to a discussion of specific models for the noise correlation function. After a brief discussion of the product correlator model that has been the basis of most earlier work on objective state vector reduction, we turned to a detailed analysis of a thermal correlation function model, in which the spatial and temporal correlations are linked by a mass-shell constraint. We showed that the dilute, nonrelativistic limit of the thermal correlator model can be put in direct correspondence with the formulae of the standard Gaussian CSL model. We then gave a detailed phenomenological analysis of thermal dark matter as the noise source, and concluded section 6 with a discussion of the behavior of thermal unparticles as the noise source, sketching qualitative behaviors for a range of values of the chemical potential and of the unparticle anomalous scaling dimension. The examples given included cases in which $\Gamma(t)$ and the energy production both are finite at $t = \infty$, and in which $\Gamma(t)$ and the energy production both grow as a fractional power smaller than unity as $t \to \infty$.

The appendices deal with various details connected with the main discussion. In appendix A, we estimate the validity of the Markovian approximation used in the energy production discussion, while in appendix B we compare the master equation used in our discussion with a more general class of master equations appearing in the literature. Appendix C gives the evaluation of integrals for the dilute, nonrelativistic model, while appendix E gives details of the unparticle correlation functions. Appendix D shows that, in a simple model, reduction is exponential in the rate function $\Gamma(t)$, indicating that the lower bound derived in section 3, as opposed to the upper bound derived there, gives the better estimate of the qualitative reduction behavior.

We can point to a number of possible directions for generalization or extension of the results of this paper. (i) We have considered only the case of a real noise coupling, corresponding to an anti-self-adjoint Hamiltonian term. More generally, one could consider a complex noise coupling, containing both real and imaginary noise couplings, with the real term contributing both to density matrix diagonalization (i.e., decoherence) and to state vector reduction, and the imaginary part contributing only to decoherence. (ii) For simplicity, we have only considered a scalar noise field ϕ, but a general treatment of non-white noises would allow for the possibility of spin-1/2 or spin-1 noise fields. Such an extension may ultimately be required on phenomenological grounds to make contact with experiment. (iii) Although we sketched the qualitative behavior of the thermal unparticle case, we did not attempt to make a quantitative phenomenological survey of the four-dimensional parameter space of this model, and this would be of interest. (iv) The derivation of lower and upper bounds on the reduction rate in section 3 made use of a positivity assumption, which is not obeyed in the thermal correlator model; can this assumption be eliminated? (v) The model calculation of appendix D indicated an exponential dependence of the reduction factor on Γ, agreeing with the corresponding density matrix diagonalization calculation but differing by a factor of 2 in the exponent from the corresponding lower bound of section 3. Can this result be generalized to the case of many particles and a wavefunction that is the superposition of many localized states as in (31)? Clearly, a general argument that the reduction factor has exponential rather than power-law dependence on Γ would be significant for the phenomenology of objective reduction models. (vi) In section 6.2 we formulated our thermal model for the correlation function, by neglecting the temperature-independent Greens function Δ_+, which reflects the non-commutativity of creation and annihilation operators. As noted, this gives an effectively classical model for the thermal noise. It would be worth exploring a fully quantum-mechanical treatment of state vector reduction by a thermal noise field, in which all parts of the quantum-mechanical correlation function (92) are retained.

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Appendix A. Markovian approximation

One can estimate the validity of the Markovian approximation by considering the case of a single free particle of mass m, so that $H = p^2/(2m)$. Then we easily calculate that

$$M(\vec{y}, s-t) = m\delta^3(\vec{y} - \mathrm{e}^{\mathrm{i}H(s-t)}\vec{q}\,\mathrm{e}^{-\mathrm{i}H(s-t)}) = m\delta^3(\vec{y} - \vec{q} - (\vec{p}/m)(s-t)), \tag{A.1}$$

so that repeating the steps leading to (83) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,\rho(t)H = \gamma m \int_0^t \mathrm{d}s \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{f(k)}{\omega_k}\cos(\omega_k(t-s))\mathrm{Tr}\,\mathcal{O}(s-t), \tag{A.2}$$

with $\mathcal{O}(s-t)$ given by

$$\mathcal{O}(s-t) = -\tfrac{1}{2}[\mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}+(\vec{p}/m)(s-t))}, [\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}}, \vec{p}^2]]. \tag{A.3}$$

This expression can be simplified by the use of the Baker–Hausdorff theorem and the canonical commutation relations, giving after considerable algebra, and dropping terms that are odd in $\vec{k}$,

$$\mathcal{O}(s-t) = k^2 \cos\left(\frac{\vec{k}\cdot\vec{p}}{m}(s-t)\right)\cos\left(\frac{k^2}{2m}(s-t)\right) - 2\vec{p}\cdot\vec{k}\sin\left(\frac{\vec{k}\cdot\vec{p}}{m}(s-t)\right)\sin\left(\frac{k^2}{2m}(s-t)\right). \tag{A.4}$$

We see that all dependence of $\mathcal{O}(s-t)$ on $s-t$ is through oscillatory terms. Assuming that the characteristic spatial variation scale of the problem is governed by $\omega_k \sim |\vec{k}| \sim |\vec{p}| \sim k_{\max}$, with $k_{\max}$ the characteristic k-value at which $f(k)$ cuts off, then when the particle mass m is large enough for the kinetic energy at $k_{\max}$ to obey

$$\frac{k_{\max}^2}{2m} \ll k_{\max}, \tag{A.5}$$

the variation of $\mathcal{O}(s-t)$ with s is much slower than that of the cosine factor in (A.2). In this case the integral in (A.2) is well approximated by replacing $\mathcal{O}(s-t)$ by $\mathcal{O}(0) = k^2$, which recovers the result of the Markovian approximation made in section 5.

Appendix B. Comparison with master equations for decoherence

A further understanding of the effect of the thermal field $\phi(\vec{x}, t)$ on the evolution of the wavefunction can be obtained by comparing (8) for the density matrix with typical master equations used for describing open quantum systems. Here we will follow the path outlined in [17], where a comparison of this kind has been made between the GRW model [18] and collisional decoherence [19–21].

We consider the evolution of a single particle; under the Markovian approximation $(M(\vec{y}, s-t) = M(\vec{y}, 0) = M(\vec{y}))$ discussed in section 5, (8) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}[H, \rho(t)] + \mathcal{L}_t^{\phi}[\rho(t)], \tag{B.1}$$

with

$$\mathcal{L}_t^{\phi}[\rho] = -\gamma \int \mathrm{d}^3x \int \mathrm{d}^3y [M(\vec{x}), [M(\vec{y}), \rho]] F(\vec{x}-\vec{y}, t), \tag{B.2}$$

and $M(\vec{x}) = m\delta^3(\vec{x}-\vec{q})$. The term $\mathcal{L}_t^{\phi}$, which includes the effect of the thermal field $\phi(\vec{x}, t)$ on $\rho(t)$, is the one we will focus on. Let us introduce the Fourier transform

$$F(\vec{x}-\vec{y}, t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \hat{F}(\vec{k}, t)\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{B.3}$$

with $\hat{F}(\vec{k}, t) = \hat{F}(-\vec{k}, t)$ due to spatial inversion invariance. One can rewrite (B.2) in terms of $\hat{F}(\vec{k}, t)$ as follows:

$$\mathcal{L}_t^{\phi}[\rho] = 2m^2\gamma \int \frac{\mathrm{d}^3k}{(2\pi)^3} \hat{F}(\vec{k}, t)[\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}}\rho\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{q}} - \rho]. \tag{B.4}$$

The above expression falls into the general class of translational-invariant Markovian master equations first given by Holevo [22] which, in the case of a bounded mapping $\mathcal{L}$, reads

$$\mathcal{L}[\rho] = \int \mathrm{d}\mu(\vec{k}) \sum_{n=1}^{\infty} \left[\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}} L_n(\vec{k}, \vec{p})\rho L_n^{\dagger}(\vec{k}, \vec{p})\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{q}} - \frac{1}{2}\{L_n^{\dagger}(\vec{k}, \vec{p})L_n(\vec{k}, \vec{p}), \rho\}\right], \tag{B.5}$$

where $L_n(\vec{k}, \vec{p})$ are bounded functions of the momentum operator $\vec{p}$ and $\mu(\vec{k})$ is a positive σ-finite measure. Briefly, the physical content of (B.5) is the following: the unitary operators

$e^{i\vec{k}\cdot\vec{q}}$ and $e^{-i\vec{k}\cdot\vec{q}}$ induce a momentum transfer to the particle by an amount equal to $\vec{k}$, while the operators $L_n(\vec{k}, \vec{p})$ imply that the momentum transfer to the particle depends on the momentum of the particle itself. This allows for mechanisms such as relaxation to take place.

Equation (B.5) reduces to (B.4) under the following circumstances. Let us assume that $L_n(\vec{k}, \vec{p}) = L_n(\vec{k})$ does not depend on the momentum $\vec{p}$ of the particle. They then become c-number functions, commuting with all other operators. By setting

$$d\mu(\vec{k}) \sum_{n=1}^{\infty} |L_n(\vec{k})|^2 = 2m^2\gamma \frac{d^3k}{(2\pi)^3} \hat{F}(\vec{k}, t), \tag{B.6}$$

the link is established. Of course, in the truly Markovian case one has $D(\vec{x} - \vec{y}, t - s) = G(\vec{x} - \vec{y})\delta(t - s)$ so that $\hat{F}(\vec{k}, t) = (1/2)\hat{G}(\vec{k})$ is independent of time, where $\hat{G}(\vec{k})$ is the Fourier transform of $G(\vec{x} - \vec{y})$.

According to the above analysis, the effect of the thermal field is not only that of localizing the wavefunction in space (this is a consequence of the specific form of the stochastic equation (13)), but also of exchanging momentum between the particle and the field. This is the reason why both the momentum and the energy of the particle are not conserved, in general. One would expect the energy of the particle to thermalize to that of the random field; however, the model described by (13) does not allow for thermalization, since the operators $L_n(\vec{k})$ do not depend on the momentum $\vec{p}$ of the particle. This is in agreement with the results of section 5.1 on energy production. The comparison with decoherence suggests how the model can be modified in order to include also such an effect; this will be a subject of future research.

Appendix C. Integrals in the dilute, nonrelativistic thermal model

From (103) we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{k^2 \mathrm{d}k}{(2\pi)^2} \mathrm{e}^{-k^2/(2\mu T)} \frac{\sin(kR)}{kR} = \left(\frac{\mu T}{2\pi}\right)^{3/2} \mathrm{e}^{-(\mu T R^2/2)}, \\
&\int_{-\infty}^{\infty} \frac{k^2 \, \mathrm{d}k}{(2\pi)^2} \mathrm{e}^{-k^2/(2\mu T)} \frac{\sin(kR)}{kR} \exp\left(\mathrm{i}\left(\mu + \frac{k^2}{2\mu}\right)t\right) \\
&\qquad = \left(\frac{\mu T}{2\pi}\right)^{3/2} (1 + t^2T^2)^{-3/4} \mathrm{e}^{-(\mu T R^2/2)/(1+t^2T^2)} \\
&\qquad \times \exp(\mathrm{i}(\mu t + (3/2)\tan^{-1}(tT) - (\mu t T^2 R^2/2)/(1 + t^2T^2))),
\end{aligned} \tag{C.1}$$

from which, by forming linear combinations, taking real and imaginary parts, and taking limits as $R = |\vec{x}| \to 0$, we get

$$\begin{aligned}
D(\vec{0}, t) - D(\vec{x}, t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu} \left(\frac{\mu T}{2\pi}\right)^{3/2} (1 + t^2T^2)^{-3/4} [\cos(\mu t + (3/2)\tan^{-1}(tT)) \\
&\quad - \mathrm{e}^{-(\mu T R^2/2)/(1+t^2T^2)} \cos(\mu t + (3/2)\tan^{-1}(tT) - (\mu t T^2 R^2/2)/(1 + t^2T^2))], \\
F(\vec{0}, t) - F(\vec{x}, t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^2} \left(\frac{\mu T}{2\pi}\right)^{3/2} (1 + t^2T^2)^{-3/4} [\sin(\mu t + (3/2)\tan^{-1}(tT)) \\
&\quad - \mathrm{e}^{-(\mu T R^2/2)/(1+t^2T^2)} \sin(\mu t + (3/2)\tan^{-1}(tT) - (\mu t T^2 R^2/2)/(1 + t^2T^2))], \\
I(\vec{0}, t) - I(\vec{x}, t) &\simeq \frac{\mathrm{e}^{-(\mu-\zeta)/T}}{\mu^3} \left(\frac{\mu T}{2\pi}\right)^{3/2} \{1 - \mathrm{e}^{-(\mu T R^2/2)} \\
&\quad - (1 + t^2T^2)^{-3/4} [\cos(\mu t + (3/2)\tan^{-1}(tT)) - \mathrm{e}^{-(\mu T R^2/2)/(1+t^2T^2)} \\
&\quad \times \cos(\mu t + (3/2)\tan^{-1}(tT) - (\mu t T^2 R^2/2)/(1 + t^2T^2))]\}.
\end{aligned} \tag{C.2}$$

Appendix D. Time evolution of the wavefunction and exponential decay of superpositions

As mentioned in the introduction, an alternative form of the collapse equation has been given in (I), which differs from (13) by a change of measure for the noise; see (35) and (37) of (I). The advantage of this alternative formulation is that it can be expressed in terms of a *linear*, but *not* norm preserving, equation (34) of (I), which is simpler to solve. Upon normalization and change of measure, one recovers the usual collapse dynamics.

Let us specialize to the case of a single particle; let us moreover set $H = 0$, as we want to focus only on the collapse mechanics. Then, for the mass density coupling considered in this paper, the linear equation reads

$$\frac{\mathrm{d}|\chi(t)\rangle}{\mathrm{d}t} = \left[\sqrt{\gamma}\int \mathrm{d}^3x\, M(\vec{x})\phi(\vec{x},t) - 2\gamma\int \mathrm{d}^3x \int \mathrm{d}^3y\, M(\vec{x})M(\vec{y})F(\vec{x}-\vec{y},t)\right]|\chi(t)\rangle. \tag{D.1}$$

The random field $\phi(\vec{x},t)$ is now supposed to be a Gaussian thermal field with respect to a new measure $\mathbb{Q}$, having mean 0 and correlator $D(\vec{x}-\vec{y},t-s)$. The relation between the statistical averages with respect to this measure and the averages with respect to the physical measure used throughout this paper (which we shall call $\mathbb{P}$ from now on) is

$$\mathbb{E}_{\mathbb{P}}[f(t)] = \mathbb{E}_{\mathbb{Q}}[f(t)\langle\chi(t)|\chi(t)\rangle], \tag{D.2}$$

where $f(t)$ is a generic random function of time.

Because of the special form (3) of the particle density operator $M(\vec{x})$, (D.1) can be readily solved in the coordinate representation $\chi(\vec{x},t) = \langle\vec{x}|\chi(t)\rangle$,

$$\chi(\vec{x},t) = \exp[\sqrt{\gamma}m\Phi(\vec{x},t) - 2\gamma m^2 I(\vec{0},t)]\chi(\vec{x},0), \tag{D.3}$$

with

$$\Phi(\vec{x},t) = \int_0^t \mathrm{d}s\phi(\vec{x},s), \qquad I(\vec{0},t) = \int_0^t \mathrm{d}s\, F(\vec{0},s) \tag{D.4}$$

($I(\vec{x},t)$ has been first introduced in (97).) Let us fix an arbitrary time t. Then the random field $\Phi(\vec{x},t)$ is a Gaussian field in the variable $\vec{x}$, with mean and correlator equal to

$$\mathbb{E}_{\mathbb{Q}}[\Phi(\vec{x},t)] = 0, \qquad \mathbb{E}_{\mathbb{Q}}[\Phi(\vec{x},t)\Phi(\vec{y},t)] = 2I(\vec{x}-\vec{y},t). \tag{D.5}$$

The above statistical properties refer to the measure $\mathbb{Q}$, while we need them to be expressed with respect to the physical measure $\mathbb{P}$. Equation (D.2) allows us to switch between the two measures, once the squared norm $\langle\chi(t)|\chi(t)\rangle$ has been computed.

In analogy with the discussion of section 3, let us consider an initial state of the form

$$\chi(\vec{x},0) = \alpha_1\delta^3(\vec{x}-\vec{r}^{\,1})^{1/2} + \alpha_2\delta^3(\vec{x}-\vec{r}^{\,2})^{1/2}, \tag{D.6}$$

corresponding to the superposition of two states well localized around $\vec{r}^{\,1}$ and $\vec{r}^{\,2}$ respectively. By substituting it into (D.3) and normalizing the wavefunction, one obtains for the collapse probabilities

$$\begin{aligned} p_1(t) = |\alpha_1(t)|^2 &= \frac{p_1\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,1},t)}}{p_1\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,1},t)} + p_2\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,2},t)}}, \\ p_2(t) = |\alpha_1(t)|^2 &= \frac{p_2\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,2},t)}}{p_1\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,1},t)} + p_2\,\mathrm{e}^{2\sqrt{\gamma}m\Phi(\vec{r}^{\,2},t)}}, \end{aligned} \tag{D.7}$$

with $p_1 = |\alpha_1|^2$ and $p_2 = |\alpha_2|^2$. Using (D.2), together with the equation

$$\langle \chi(t)|\chi(t)\rangle = p_1 \exp[2\sqrt{\gamma} m \Phi(\vec{r}^{\,1}, t) - 4\gamma m^2 I(\vec{0}, t)] + p_2 \exp[2\sqrt{\gamma} m \Phi(\vec{r}^{\,2}, t) - 4\gamma m^2 I(\vec{0}, t)], \tag{D.8}$$

we can compute the average of the product $p_1(t)p_2(t)$.

Due to the statistical properties (D.5), the joint probability density of the two random variables $\Phi(\vec{r}^{\,1}, t)$ and $\Phi(\vec{r}^{\,2}, t)$ reads

$$P_{12}^{\mathbb{Q}} = \frac{1}{2\pi\sqrt{a_t^2 - b_t^2}} \exp\left[-\frac{a_t(\Phi(\vec{r}^{\,1}, t))^2 - 2a_t b_t \Phi(\vec{r}^{\,1}, t)\Phi(\vec{r}^{\,2}, t) + a_t(\Phi(\vec{r}^{\,2}, t))^2}{2(a_t^2 - b_t^2)} \right], \tag{D.9}$$

with $a_t = 2I(\vec{0}, t)$ and $b_t = 2I(\vec{r}^{\,1} - \vec{r}^{\,2}, t)$. Using now (D.2), (D.7), (D.8) and (D.9) we get

$$\mathbb{E}_{\mathbb{P}}[p_1(t)p_2(t)] = p_1 p_2\, \mathrm{e}^{-2\gamma m^2 a_t} \frac{1}{8\pi\gamma m^2 a_t \sqrt{1 - r_t^2}} \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{d}y \frac{\exp\left[-\frac{x^2 + y^2 - 2r_t x y}{8\gamma m^2 a_t (1 - r_t^2)} + x + y \right]}{p_1\, \mathrm{e}^x + p_2\, \mathrm{e}^y}, \tag{D.10}$$

with $r_t = b_t/a_t$; we have also relabeled $x = 2\sqrt{\gamma} m \Phi(\vec{r}^{\,1}, t)$ and $y = 2\sqrt{\gamma} m \Phi(\vec{r}^{\,2}, t)$. Equation (D.10) can be further simplified by making the change of variables $t = (x + y)/2$, $s = (x - y)/2$. In this case, the two integrals decouple and one gets

$$\mathbb{E}_{\mathbb{P}}[p_1(t)p_2(t)] = p_1 p_2\, \mathrm{e}^{-\Gamma(t)} \frac{1}{2\sqrt{\pi\Gamma(t)}} \int_{-\infty}^{+\infty} \mathrm{d}s \frac{\mathrm{e}^{-s^2/4\Gamma(t)}}{p_1\, \mathrm{e}^s + p_2\, \mathrm{e}^{-s}}, \tag{D.11}$$

where $\Gamma(t) = \gamma m^2 a_t (1 - r_t)$ corresponds to definition (19). The final integral gives a finite contribution as $\Gamma(t) \to \infty$, which proves that the decay of the superposition is exponential in time, and proportional to $\mathrm{e}^{-\Gamma(t)}/\sqrt{\Gamma(t)}$. In particular, by using the inequality

$$p_1\, \mathrm{e}^s + p_2\, \mathrm{e}^{-s} \geqslant \overline{m}(\mathrm{e}^s + \mathrm{e}^{-s}) = 2\overline{m} \cosh s, \tag{D.12}$$

with $\overline{m} \equiv \min\{p_1, p_2\}$ (here we assume that $\overline{m} \neq 0$; the trivial case $\overline{m} = 0$ can be treated separately), one has

$$\int_{-\infty}^{+\infty} \mathrm{d}s \frac{\mathrm{e}^{-s^2/4\Gamma(t)}}{p_1\, \mathrm{e}^s + p_2\, \mathrm{e}^{-s}} \leqslant \frac{1}{2\overline{m}} \int_{-\infty}^{+\infty} \mathrm{d}s \frac{1}{\cosh s} = \frac{\pi}{2\overline{m}}. \tag{D.13}$$

Collecting all results, we can write

$$\mathbb{E}_{\mathbb{P}}[p_1(t)p_2(t)] \leqslant \mathbb{E}_{\mathbb{P}}[p_1(0)p_2(0)] \frac{\sqrt{\pi}}{4\overline{m}\sqrt{\Gamma(t)}}\, \mathrm{e}^{-\Gamma(t)}. \tag{D.14}$$

Appendix E. Unparticle thermal correlation functions

We take the unparticle thermal correlation function to be given by an average over thermal correlation functions for particles of mass $\mu \geqslant 0$, using the same weighting function $\rho(\mu^2)$ that is used [25] to generate the unparticle propagator from the propagator for a boson of mass μ,

$$\rho(\mu^2) = (d - 1)\Lambda^{2(1-d)}(\mu^2)^{d-2}. \tag{E.1}$$

Writing the left-hand side of (94) as $D(\vec{x}, t, \mu)$ so as to explicitly show the mass dependence, the thermal unparticle correlation function $D_{\mathcal{U}}$ is then given by

$$D_{\mathcal{U}}(\vec{x}, t) = \int_0^{\infty} \mathrm{d}\mu^2 \rho(\mu^2) D(\vec{x}, t, \mu). \tag{E.2}$$

Substituting (94) and (95), we thus get

$$D_{\mathcal{U}}(\vec{x},t) = (d-1)\Lambda^{2(1-d)}\int_0^\infty \mathrm{d}\mu^2(\mu^2)^{d-2}\int \frac{\mathrm{d}^3k}{(2\pi)^3\omega_k}\frac{1}{\mathrm{e}^{\frac{\omega_k-\zeta}{T}}-1}\cos(\vec{k}\cdot\vec{x})\cos(\omega_k t)$$
$$= (d-1)\Lambda^{2(1-d)}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot\vec{x})\int_0^\infty \mathrm{d}\mu^2(\mu^2)^{d-2}\frac{1}{\omega_k}\frac{1}{\mathrm{e}^{\frac{\omega_k-\zeta}{T}}-1}\cos(\omega_k t), \tag{E.3}$$

where in the second line we have isolated those factors of the integrand that explicitly depend on μ. Since $\omega_k^2 = k^2+\mu^2$, we can change integration variable in the inner integral from μ^2 to ω_k^2, by using

$$d\mu^2 = 2\omega_k d\omega_k, \qquad (\mu^2)^{d-2} = \left(\omega_k^2-k^2\right)^{d-2}, \tag{E.4}$$

which gives

$$\int_0^\infty \mathrm{d}\mu^2(\mu^2)^{d-2}\frac{1}{\omega_k}\frac{1}{\mathrm{e}^{\frac{\omega_k-\zeta}{T}}-1}\cos(\omega_k t) = 2\int_k^\infty \mathrm{d}\omega\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\cos(\omega t), \tag{E.5}$$

where we have relabeled the dummy integration variable ω_k as ω. Substituting this into (E.3) we get

$$D_{\mathcal{U}}(\vec{x},t) = 2(d-1)\Lambda^{2(1-d)}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot\vec{x})\int_k^\infty \mathrm{d}\omega\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\cos(\omega t). \tag{E.6}$$

Corresponding to this formula, the function $F_{\mathcal{U}}(\vec{x},t)$ introduced in (11) is given by

$$F_{\mathcal{U}}(\vec{x},t) = \int_0^t \mathrm{d}s\, D_{\mathcal{U}}(\vec{x},t-s)$$
$$= 2(d-1)\Lambda^{2(1-d)}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot\vec{x})\int_k^\infty \frac{\mathrm{d}\omega}{\omega}\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\sin(\omega t), \tag{E.7}$$

and the integral appearing in the rate function $\Gamma(t)$ of (17) is given by

$$I_{\mathcal{U}}(\vec{x},t) \equiv \int_0^t \mathrm{d}s\, F_{\mathcal{U}}(\vec{x},s)$$
$$= 2(d-1)\Lambda^{2(1-d)}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos(\vec{k}\cdot\vec{x})\int_k^\infty \frac{\mathrm{d}\omega}{\omega^2}\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}[1-\cos(\omega t)]. \tag{E.8}$$

From (E.7) we can read off the Fourier transform defined in (70), from which the energy production is calculated through (71),

$$\hat{F}_{\mathcal{U}}(\vec{k},t) = 2(d-1)\Lambda^{2(1-d)}\int_k^\infty \frac{\mathrm{d}\omega}{\omega}\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\sin(\omega t). \tag{E.9}$$

Note that in all of these formulae, the scale parameter Λ appears as an overall factor, which then combines with the noise coupling γ to give a new effective coupling $\gamma\Lambda^{2(1-d)}$.

The correlation function $D_{\mathcal{U}}(\vec{x},t)$ can be written in several alternative forms. Performing the angular average over $\vec{k}$, we get

$$D_{\mathcal{U}}(\vec{x},t) = (d-1)\Lambda^{2(1-d)}\int_0^\infty \frac{\mathrm{d}k\,k}{\pi^2|\vec{x}|}\sin(k|\vec{x}|)\int_k^\infty \mathrm{d}\omega\frac{(\omega^2-k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\cos(\omega t), \tag{E.10}$$

which on interchange of orders of the k and ω integrations becomes

$$D_{\mathcal{U}}(\vec{x},t) = (d-1)\Lambda^{2(1-d)}\int_0^\infty \mathrm{d}\omega\frac{\cos(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}}-1}\int_0^\omega \frac{\mathrm{d}k\,k}{\pi^2|\vec{x}|}\sin(k|\vec{x}|)(\omega^2-k^2)^{d-2}. \tag{E.11}$$

Making the change of integration variable $k = \omega v$, this can be further rewritten as

$$D_{\mathcal{U}}(\vec{x}, t) = (d-1)\Lambda^{2(1-d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2(d-1)}\cos(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \int_0^1 \frac{\mathrm{d}v\, v}{\pi^2 |\vec{x}|} \sin(v\omega|\vec{x}|)(1-v^2)^{d-2}. \tag{E.12}$$

The integral over v in (E.12) converges only for $\mathrm{Re}\, d > 1$. However, by an integration by parts this integral is transformed as follows:

$$\int_0^1 \frac{\mathrm{d}v v}{\pi^2|\vec{x}|} \sin(v\omega|\vec{x}|)(1-v^2)^{d-2} = \int_0^1 \frac{\mathrm{d}v\, \omega}{2\pi^2(d-1)} \cos(v\omega|\vec{x}|)(1-v^2)^{d-1}, \tag{E.13}$$

which gives an analytic continuation around the simple pole at $d = 1$, expressed in terms of a v integral that now converges for $\mathrm{Re}\, d > 0$. Substituting (E.13) into (E.12) gives a formula for the correlation function which is now manifestly finite for $\mathrm{Re}\, d > 0$,

$$D_{\mathcal{U}}(\vec{x}, t) = \frac{1}{2}\Lambda^{2(1-d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-1}\cos(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \int_0^1 \frac{\mathrm{d}v}{\pi^2} \cos(v\omega|\vec{x}|)(1-v^2)^{d-1}. \tag{E.14}$$

The corresponding formulae for $F_{\mathcal{U}}(\vec{x}, t)$ and $I_{\mathcal{U}}(\vec{x}, t)$ are now obtained by the replacement of $\cos(\omega t)$ by $\sin(\omega t)/\omega$ and $[1 - \cos(\omega t)]/\omega^2$, respectively,

$$F_{\mathcal{U}}(\vec{x}, t) = \frac{1}{2}\Lambda^{2(1-d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-2}\sin(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \int_0^1 \frac{\mathrm{d}v}{\pi^2} \cos(v\omega|\vec{x}|)(1-v^2)^{d-1} \tag{E.15}$$

and

$$I_{\mathcal{U}}(\vec{x}, t) = \frac{1}{2}\Lambda^{2(1-d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-3}[1-\cos(\omega t)]}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \int_0^1 \frac{\mathrm{d}v}{\pi^2} \cos(v\omega|\vec{x}|)(1-v^2)^{d-1}. \tag{E.16}$$

From (E.16) we find for the subtracted integral that enters into $\Gamma(t)$,

$$I_{\mathcal{U}}(\vec{0}, t) - I_{\mathcal{U}}(\vec{x}, t) = \frac{1}{2}\Lambda^{2(1-d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-3}[1-\cos(\omega t)]}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \times \int_0^1 \frac{\mathrm{d}v}{\pi^2}[1-\cos(v\omega|\vec{x}|)](1-v^2)^{d-1}, \tag{E.17}$$

giving an expression that is manifestly positive.

Let us now return to (E.9), and use it to calculate the energy production. Substituting (E.9) into (71) we get, for a single particle with mass-coupled unparticle noise,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, H\rho(t) = \frac{\gamma m \Lambda^{2(1-d)}(d-1)}{\pi^2} \int_0^\infty \mathrm{d}k\, k^4 \int_k^\infty \frac{\mathrm{d}\omega}{\omega} \frac{(\omega^2 - k^2)^{d-2}}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \sin(\omega t), \tag{E.18}$$

which on interchanging the orders of the k and ω integrations becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, H\rho(t) = \frac{\gamma m \Lambda^{2(1-d)}(d-1)}{\pi^2} \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \frac{\sin(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1} \int_0^\omega \mathrm{d}k\, k^4(\omega^2 - k^2)^{d-2}. \tag{E.19}$$

Making the change of variable $k = \omega u^{1/2}$, $\mathrm{d}k = (1/2)\omega u^{-1/2}\mathrm{d}u$ in the inner integral, it can be evaluated in terms of the Euler B function; then using $(d-1)\Gamma(d-1) = \Gamma(d)$ one gets the compact expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, H\rho(t) = \frac{3\gamma m \Lambda^{2(1-d)}}{(2\pi)^2} \frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d}\sin(\omega t)}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1}, \tag{E.20}$$

with the integral convergent for $\mathrm{Re}\, d > 0$. Integrating over t, one gets the corresponding formula for the total energy production

$$\mathrm{Tr}\, H\rho(t) - \mathrm{Tr}\, H\rho(0) = \frac{3\gamma m \Lambda^{2(1-d)}}{(2\pi)^2} \frac{\Gamma(3/2)\Gamma(d)}{\Gamma(3/2+d)} \int_0^\infty \mathrm{d}\omega \frac{\omega^{2d-1}[1-\cos(\omega t)]}{\mathrm{e}^{\frac{\omega-\zeta}{T}} - 1}. \tag{E.21}$$

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